1. A set of \( N \) independent electrical dipoles, each with a dipole moment of \( \mu \), occupy a volume \( V \). An electrical field \( \vec{E} \) is applied.

(a) Calculate the electrical polarization \( \vec{P}(T, \vec{E}) \) of the system at temperature \( T \). The polarization of the system is defined as the net dipole moment per unit volume.

(b) Find the dielectric susceptibility of the system \( \chi(T) = \frac{1}{\epsilon_0} \left( \frac{\partial P}{\partial E} \right)_T \) for weak electric fields \( \vec{E} \rightarrow 0 \).

(c) Show that the root mean square fluctuation of the polarization \( \vec{P} \) in the absence of an electric field can be simply related to \( \chi \).

(d) Calculate the dielectric constant \( \kappa = 1 + \chi \) for ice at 0°C, assuming as above, that the dipole moments of the water molecules do not interact with one another (a drastic assumption). You are given that the dipole moment of a single \( \text{H}_2\text{O} \) molecule is \( \mu = 6.2 \times 10^{-30} \) Coulomb-meter. (The density of ice is \( 0.9 \times 10^3 \) kg/m\(^3\) and the permittivity of free space is \( \epsilon_0 = 8.85 \times 10^{-12} \) in SI units).

2. Consider a classical gas composed of \( N \) particles of mass \( m \) occupying a box of volume \( V \) at temperature \( T \). The particles interact pairwise via the interparticle potential:

\[
v(r) = \frac{A}{r^n}
\]

where \( A \) and \( n \) are positive quantities.

(a) We would like to ensure that the system actually possesses a sensible thermodynamic limit - where, for example, energy is an extensive quantity. This requires that the potential \( v \) decay sufficiently rapidly. Find the resulting condition on \( n \).

(b) In principle, using the Hamiltonian:

\[
H = \sum_{i=1}^{N} \frac{\vec{p}_i^2}{2m} + \sum_{i<j} v(\left| \vec{r}_i - \vec{r}_j \right|)
\]
one could compute the pressure \( p \) and internal energy \( E \) in terms of the thermodynamic variables \( N, V, T \), but this is not an easy task. Assume therefore that you have been given \( p \) (in terms of \( N, V, T \)). Find a relation between the energy \( E \) and \( p, N, V, T \). \textit{(Hint: Use a scaling procedure to relate the potential energy to the pressure)}