1. Bose Einstein condensation of a dilute gas of atoms of mass $m$ is achieved in the laboratory by confining them in a magnetic trap that gives rise to a harmonic potential, so that the Hamiltonian for a single particle in the trap takes the form:

$$H = \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + \frac{m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$$ (1)

where $\omega_x$, $\omega_y$, $\omega_z$ represent the trap frequencies in the different directions. Let $\bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3}$.

(a) For large enough energies $\epsilon \gg \hbar \bar{\omega}$ the discrete energy eigenvalues may be approximated by a continuous distribution with a density of states $N(\epsilon)$, where $N(\epsilon) \Delta \epsilon$ is the number of energy eigenstates in the energy interval $(\epsilon, \epsilon + \Delta \epsilon)$. Find an expression for $N(\epsilon)$.

(b) Using this approximation derive an equation for the fugacity $z$ of the bosons, given that there are a total of $N$ atoms in the trap. Use this to obtain the Bose-Einstein condensation temperature $T_{BEC}$. Explain the physical interpretation of this result. What is the condition that needs to be satisfied for the continuous density of states approximation to yield an accurate value for $T_{BEC}$?

(c) Find an expression for the condensate density at a temperature $T < T_{BEC}$ using the continuous density of states approximation.

(d) The BEC setup in Prof. Stamper-Kurn’s laboratory typically uses Rb$_{87}$ atoms confined in a trap with frequencies $\frac{1}{2\pi} (\omega_x, \omega_y, \omega_z) = (250, 670, 7)$ Hertz and $N = 10^7$ atoms. Find $T_{BEC}$ for this case. You may use the fact that $1 + \frac{1}{2} + \frac{1}{3} + \ldots = 1.202\ldots$

2. Consider the general problem of non-interacting bosons in free space with an energy dispersion $\epsilon(p) = Ap^s$ in $d$ dimensions. (In class we considered the special case of $s = 2$ and $d = 3$). Find the condition on $s, d$ for Bose condensation to occur at a finite temperature. Does Bose condensation occur for $s = 2$ and $d = 2$?