Low energy excitations of fractional quantum Hall states are described by Landau–Ginzburg theory of superfluid coupled with a vector potential whose effective action includes a Chern–Simons term. In [1], Susskind suggested that a better description of these states is provided by a non-commutative Chern–Simons theory with gauge group $U(1)$. This idea was refined by Polychronakos [2], who proposed a modified version of this theory, and showed that quantization of the theory naturally leads to quantization of inverse filling fraction and fractional charge of quasi-hole excitation. We briefly summarize their main ideas.

**Non-commutative Chern–Simons Theory**

To motivate the idea, let us start with an alternative derivation of Chern–Simons theory as low energy effective description of fractional quantum Hall states. The Lagrangian of $N$ electrons moving on a plane with external magnetic field $B$ is

$$L = \sum_{a=1}^{N} \left\{ \frac{1}{2} m_e \dot{x}_a^2 - \frac{e}{2} B_\epsilon x_a^i \dot{x}_a^j \right\} - U(\vec{x}).$$

As we pass to effective continuum description, we replace the particle label $a$ by two-dimensional coordinate $y = (y_1, y_2)$. These coordinates label points in the fluid and move with it (the “comoving coordinates”), and their physical coordinates are given by a pair of fields $x_i(y, t)$, with $i = 1, 2$, defined on $y$-space.

We choose $y$-coordinates so that the density $\rho_0$ is constant in $y$-space. The physical density is then given by multiplying $\rho_0$ with the Jacobian of the map $x_i(y, t)$. The Lagrangian becomes

$$L = \rho_0 \int d^2y \left[ \frac{m_e}{2} \dot{x}_i^2 - \frac{e}{2} B_\epsilon x_i^i \dot{x}_j^j - V \left( \rho_0 \left| \frac{\partial y}{\partial x} \right| \right) \right].$$

Assuming the potential $V$ is of the form $V = \mu (\rho - \rho_0)^2$, there is an obvious solution of the equations of motion: $x_i(y, t) = y_i$. To describe the small fluctuations of the fluid from this equilibrium state, one introduces a vector field $A_i$ as follows:

$$x_i = y_i + \frac{\epsilon_{ij} A_j}{2\pi \rho_0}.$$
To linear order in $A$, the Lagrangian then becomes

$$L = \frac{1}{2g^2} \int d^2y \left[ \dot{A}^2 - \frac{2\mu m_0^2}{m_e} (\nabla \times A)^2 - \frac{eB}{m_e} \epsilon_{ij} A_i \dot{A}_j \right],$$

where $g^2 = \frac{4\pi^2 \rho_0}{m_e}$.

The first two terms are those of Maxwell Lagrangian in the temporal gauge $A_0 = 0$, and the third is the Chern–Simons term. Furthermore, the theory is clearly invariant under area-preserving diffeomorphisms of $y$-space, and this translates into $U(1)$ gauge invariance of $A_i$: $\delta A_i = \partial \Lambda / \partial y_i$. One can recover the Lorentz-invariant Lagrangian by introducing $A_0$ as Lagrange multiplier to enforce the constraint imposed by gauge invariance. In three dimensions, the long distance physics is dominated by the Chern-Simons term, so we can drop the Maxwell terms.

This effective Chern–Simons theory can be improved in two ways. First, the Lagrangian which incorporates full nonlinear equations of motion takes the form

$$L = \epsilon^{\mu
u\lambda} \left( \partial_\lambda A_\mu - \frac{\theta}{3} (A_\mu, A_\nu) \right) A_\nu,$$

with corresponding nonlinear gauge transformation $\delta A_i = \partial_i \Lambda + \theta \{ A_i, \Lambda \}$. Here, $\{ \cdot, \cdot \}$ is the Poisson bracket defined on $y$-space. Second, to take into account the discrete nature of electron system, one wants to “discretize” the $y$-space with commutation relation $[y_1, y_2] = i\theta$. This leads us to the non-commutative version of Chern–Simons theory, whose Lagrangian is

$$L = \epsilon^{\mu
u\lambda} \left( A_\mu \star \partial_\lambda + \frac{2i}{3} A_\mu \star A_\nu \star A_\lambda \right),$$

where $\star$ is a deformation of usual commutative product of functions on $y$-space with non-commutativity parameter $\theta$. In fact, the nonlinear Lagrangian above can be obtained by expanding the non-commutative Chern–Simons Lagrangian to first order in $\theta$.

This theory is equivalent to the matrix theory representation where one replaces the coordinates $x_i^a$ of electrons with $N \times N$ hermitian matrices $X_i$:

$$L = \frac{eB}{2} \text{Tr} \{ e_{ij} (\dot{X}_i - i [X_i, A_0] ) X_j + 2i A_0 \}.$$

This theory has $U(N)$ gauge invariance, which is the analog of area-preserving diffeomorphisms in the previous description.

**Finite Matrix Model**

This matrix theory has classical solutions only if $N$ is infinite. To obtain a theory with finite number of electrons, one makes the following modification:

$$L = \frac{eB}{2} \text{Tr} \{ e_{ij} (\dot{X}_i - i [X_i, A_0] ) X_j + 2i A_0 - \omega X_i X_i \} + \Phi^\dagger (i \dot{\Phi} - A_0 \Phi),$$

where $\Phi$ is a complex boson in the fundamental representation of $U(N)$. Solving $A_0$ equation of motion yields the constraint

$$G = -ieB [X_1, X_2] + \Phi \Phi^\dagger - eB \theta = 0.$$
To solve this constraint and other equations of motion, one introduces an \( N \)-dimensional auxiliary Hilbert space on which the \( N \times N \) matrices act. Choosing a basis \( \{|n\rangle\}_{n=0}^{N-1} \) of this Hilbert space, the ground state of the theory is written as

\[
X_1 + iX_2 = Ae^{\text{i}t}, \quad \text{where} \quad A = \sqrt{2} \sum_{n=0}^{N-1} \sqrt{n}|n - 1\rangle\langle n|.
\]

This state has natural physical interpretation of uniform distribution of electrons inside a disk with density \( \rho_0 = 1/2\pi\theta \). One can also identify the quasihole states by choosing

\[
A = \sqrt{2\theta}\left(\sqrt{q}|N-1\rangle\langle 0| + \sum_{n=1}^{N-1} \sqrt{n+q}|n-1\rangle\langle n|\right),
\]

which can be interpreted as having a circular hole of area \( 2\pi q\theta \) at the center.

The quantization of this model is straightforward; the canonical conjugate variables to \( (X_1)_{mn} \) and \( \Phi_m \) are \( eB(X_2)_{nm} \) and \( i\Phi^\dagger \), respectively, and one obtains the Hamiltonian of \( N(N+1) \) independent oscillators. These oscillators are related by the constraint above. Its quantum version puts the following constraints on physical states of the theory:

\[
(G_X^a + G_\Phi^a)|\text{phys}\rangle = 0, \quad \left( \sum_{m=1}^{N} \Phi^\dagger_m \Phi_m - eNB\theta \right)|\text{phys}\rangle = 0.
\]

Here, the \( G_X^a \) and \( G_\Phi^a \) are the representations of \( SU(N) \) algebra on the Fock spaces constructed by \( X \) and \( \Phi \) oscillators, respectively. A little group theory analysis shows that these two constraints imply the quantization of inverse filling fraction \( m = 1/\nu \).

One can also identify all quantum states of the theory. The energy spectrum is determined by \( N \) integers \( n_j \) via the formula

\[
E = \sum_{j=1}^{N} \omega(n_j + \frac{1}{2}),
\]

with constraint \( n_j - n_{j-1} \geq m \). The ground state then corresponds to \( n_j = m(j-1) \), which is interpreted as a state with uniform density and filling fraction \( \nu = 1/m \). The quasihole state corresponds to

\[
n_j = \begin{cases} 
  n(j-1) & \text{for } j \leq k \\
  n(j-1) + 1 & \text{for } k < j \leq N,
\end{cases}
\]

for some \( k \). By comparing this state with one obtained by removing one electron corresponding to \( n_k \), one sees that quasihole has particle number \( q = -\nu \).

References
