1. Consider Ising spins on the sites $r$ of a 2D square lattice at zero temperature, which only interact via a four spin interaction on each plaquette:

$$\hat{H} = -K \sum_{r} \sigma_{r}^{z} \sigma_{r+\hat{x}}^{z} \sigma_{r+y}^{z} \sigma_{r+\hat{x}+\hat{y}}^{z} - h \sum_{r} \sigma_{r}^{x}$$

(1)

where $K > 0$ and $h$ is the strength of a transverse field term that introduces quantum dynamics into the model. (Note, in contrast to the Ising gauge theory where such four spin interactions occur between spins that live on the bond centers, here spins reside on lattice sites). We would like to study the phases of this model in the thermodynamic limit.

(a) What are the symmetries of the Hamiltonian? Write down the operators that generate the spin symmetry transformations.

(b) By studying the extreme limits of $h/K$, establish the phases of this system. Find an operator whose correlation function can distinguish between the two phases. What are the ground state degeneracies of the system with $L_x \times L_y$ sites and periodic boundary conditions?

(c) A dual formulation of the model is obtained by defining plaquette operators $\tau_p^x = \prod \sigma_i^x$ where the product is taken around the four spins of the plaquette. Complete this duality - what information can be obtained from it about the model?

(a) Consider antiferromagnetically coupled classical XY spins on the triangular lattice (sites $r$) at an inverse temperature $\beta$:

$$Z = \prod_r \int_{-\pi}^{\pi} d\phi_r e^{-\beta\mathcal{H}(\phi)}$$

$$\beta\mathcal{H} = \frac{1}{2} \sum_{r \rightarrow r'} K_{rr'} \cos(\phi_r - \phi_{r'})$$

where $K_{rr'} = K > 0$ if $r$, $r'$ are nearest neighbors, $K_{rr} = -zK$, [where $z$ is the coordination number ($z = 6$ here)] and zero otherwise. Show that this makes $-K_{rr'}$ a positive semidefinite matrix (has only non-negative eigenvalues).
(b) What are the phases of this model in the $K \to \infty$ and $K \to 0$ limits? What are the ground state configurations in the $K = \infty$ limit?

(c) Use the Hubbard Stratonovich transformation
\[
\frac{1}{\sqrt{\text{Det}[a]}} e^{\frac{1}{2} b_i \{a^{-1}\}_{ij} b_j} = \prod_i \int_{-\infty}^{\infty} \frac{dx_i}{\sqrt{\pi}} e^{-\frac{1}{2} x_i a_{ij} x_j + b_i x_i},
\]
where $a$ is a positive definite matrix, to obtain an exact rewriting of the original partition function in terms of a complex field $\psi_r$.

(d) Now obtain a low energy effective theory for $\psi_r$ retaining only the lowest energy fourier modes and assuming the magnitude of $\psi$ is small.

(e) We would like to know how the system accomplishes the transition between the $K \to 0$ and $K \to \infty$ limits. Use the effective theory obtained above and your knowledge of phases and phase transitions in $D = 2$ to describe possible scenarios. Sketch the corresponding phase diagrams and label the phases and transitions.

(f) Not for credit Perform the same analysis for the fully frustrated XY model on the square lattice. Here, the nearest neighbor bonds are such that their product around each plaquette is a negative constant.

(a) Block spin renormalization group for the triangular lattice Ising ferromagnet. Consider a triangular lattice classical Ising ferromagnet with nearest neighbor interactions:
\[
\beta E = -K \sum_{<ij>} \sigma_i \sigma_j
\]
We would like to study the transition within a real space renormalization group scheme and compare it against the exact solution.

(b) The transformation suggested consists of blocking three spins in non overlapping triangles as shown in the figure 1, and replacing them by a single spin obtained from the majority rule for spins within the triangle. In general, after one blocking transformation, the interaction does not remain nearest neighbor. However, for the purpose of the current problem, assume that it does, and calculate the effective nearest neighbor coupling $K'$ after one transformation. You may assume that interactions between different blocks of spins can be treated to lowest order in perturbation theory. From this obtain the value of the critical coupling $K^*$ and compare against the exact value $K^* = \frac{1}{4} \log 3$.

(c) By studying the change of a small deviation $\Delta K$ under blocking $\Delta K' = b^{\nu} \Delta K$ (where $b$ is the rescaling factor) obtain $\nu = 1/y_t$ and compare against the exact solution that we found $\nu = 1$.  

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(d) By introducing an external field $h$, calculate again to lowest order in perturbation theory the growth of this quantity $h' = b y_h h$. Compare against the exact solution $y_h = 15/8$.

(e) *not for credit* Can you improve on this scheme?