An Introduction to Quasiparticle Physics via a Semiclassical Approach

Ashvin Vishwanath
Department of Physics, Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544.
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The purpose of this article is twofold. First, to provide a pedagogical introduction to some of the rich physics of superconductor quasiparticles. And second, to introduce a semiclassical approximation that gives some insight into the unusual behavior of quasiparticles. We focus on the properties of quasiparticles in the presence of topological defects such as vortices, domain walls or sample edges, especially in superconductors with unconventional pairing. The Bogoliubov-deGennes equation that governs the superconductor quasiparticles are solved in a variety of such situations. We review the interesting consequences of these solutions, some of which are well known and established, while others are more recent, exotic developments. Where appropriate, the relevant experimental situation is also discussed. Subsequently, a semiclassical approximation to the Bogoliubov-deGennes equation is developed and, as an application, many of the earlier results are rederived in a simple fashion. Weak localization of superconductor quasiparticles is briefly discussed from this semiclassical viewpoint.

I. INTRODUCTION

In recent times there has been a revival of interest in the physics of superconductor quasiparticles. A strong impetus was provided by the discovery of materials that possess pairing states that are fundamentally different from the s-wave spin-singlet variety found in conventional superconductors. Although spin triplet pairing states in superfluid He₃ have been known for several decades¹, and has given rise to a rich literature, it was the recent discovery of d-wave pairing in the high temperature cuprate superconductors that reigned this field. Besides the cuprates, unconventional pairing is also believed to occur in the superconducting phases of Sr₂RuO₄, several heavy fermion compounds such as UP₃₃, and organic materials like (TMTSF)₂PF₆. Another reason for the renewed interest in this area is the development of new or improved experimental techniques such as Scanning Tunneling Microscopy (STM) and thermal transport measurements that can probe quasiparticle properties in fine detail. For example, STM measurements of the vortex core of conventional superconductors² led to the direct verification of vortex quasiparticle states that had been predicted several years back³.

The availability of unconventional pairing superconductors makes it possible to realize localization physics beyond the standard three Dyson universality classes⁴. While in the case of electrons, the localization physics falls into one of three classes depending on the presence of time reversal and spin rotation symmetry, for the case of superconductor quasiparticles, additional symmetry classes arise. If the quasiparticles are assumed to be non-interacting, then they are described by a wave equation called the Bogoliubov-deGennes equation. The exact particle hole symmetry of these equations lead to four new localization classes depending, once again, on the presence of spin rotation and time reversal symmetry. There has been much theoretical effort in recent times exploring these new localization classes.

The physics of superconductor quasiparticles has been found to be relevant in seemingly unrelated problems as well. In two dimensional electron systems in a magnetic field, a quantized Hall effect is seen at the filling fraction \( \nu = 5/2 \).⁵ One of the candidate states to account for this, which has been receiving increasing experimental and numerical support, is the Read-Moore state⁶,⁷. Briefly, this state can be understood as follows. For electrons in a half filled Landau level, attaching two statistical flux quanta per electron leads to a fluid of composite fermions in zero effective field. These can then pair and condense leading to a quantized Hall state. The Read-Moore state is obtained by a particular pairing of composite-fermions rather like that in the A¹ phase of liquid He₃. An analysis of the quasiparticle physics directly leads to the exotic properties of this state, such as excitations that obey non-Abelian statistics, and a chiral Majorana mode at the edge⁸,⁹.

Since the superconductor may (in a sense) be considered a broken symmetry state, the order parameter field can contain topological defects like vortices, and in the case of more complex pairing, domain walls and textures. Particularly interesting is the physics of quasiparticles in the presence of such topological defects or near the edge of the system. For example, we have already mentioned the vortex core states that have been seen in STM measurements of conventional superconductors. When unconventional pairing situations are considered, exotic effects, including quasiparticle fractionalization can occur. For example in chiral superconductors (where, roughly speaking, the Cooper pairs can be thought of as being in an angular momentum state that causes them to spin around) such as with \( d_{x^2-y^2} + i d_{xy} \) or \( p_x + ip_y \) pairing, an analogue of the Quantized Hall Effect is realised¹⁰,¹¹. However, since the charge on the quasiparticles is not conserved, what is quantized is the spin or thermal Hall conductance. The edge of such a two dimensional system has chiral quasiparticle modes propagating along it. Quasiparticle fractionalization can occur, for example, in some triplet superconductors (where the Cooper pairs are in a state with total spin equal to one). This is analogous...
to electron fractionalization at domain walls in charge density wave systems, that has been extensively studied in the context of polyacetylene\textsuperscript{12}. For example in a two dimensional \( p_x + ip_y \) superconductor that has one component of the quasiparticle spin (call it \( S_z \)) conserved, the vortices are found to carry \( S_z = \pm \frac{1}{2} \). For the spin polarised case of such a superconductor, there is a Majorana fermion mode at zero energy associated with the vortex\textsuperscript{8}. This endows the vortices with an exotic generalization of statistics called non-Abelian statistics. Such fractionalized states have been proposed as ideal hard-ware for the qubits of a quantum computer since they are relatively immune to error\textsuperscript{13}.

In this article we will focus on the physics of quasiparticles in the presence of defects like vortices, domain walls and sample edges, in a variety of pairing situations. For the most part, we ignore quasiparticle interactions (and self consistency in particular), and simply solve for the quasiparticle states in the presence of an externally specified pair potential. The layout of the rest of this article is as follows. In Section II, we derive the Bogoliubov-deGennes wave-equation that describes superconductor quasiparticles. We then review its solution in several different situations where they lead to interesting consequences. In view of the pedagogical nature of this article, we present these solutions in full detail. Although these situations may appear rather different from one another, we find that their solutions can all be obtained using essentially the same technique. In fact the same approach can be used in the study of quasiparticles in the mixed state of a d-wave superconductor, which is described in\textsuperscript{14} and references therein. The interesting physics associated with these situations as well as possible experimental realizations are discussed. After this review, in Section III we introduce a semiclassical approximation for quasiparticles in order to gain some insight into their dynamics. This is done by first deriving a classical mechanics for quasiparticles. In this limit the quasiparticle is described, in addition to its position and momentum, by a unit vector that captures the electron-hole mixing character of the quasiparticle. When supplemented with a quantization condition, the semiclassical formalism allows us to revisit the situations considered in Section II. Amusingly, this simple formalism obtains for us the exact results in a number of special cases. In contrast to some other semiclassical approaches in the literature on superconductor quasiparticles, this approach we believe, is physically transparent and provides some intuition for the unusual behavior of quasiparticles. This is followed by a brief discussion of weak localization of quasiparticles, from a semiclassical viewpoint. Finally, in Section IV we survey the unsolved problems in this area and list promising directions for future work.

II. QUASIPARTICLES IN VORТИCES AND AT THE EDGE: A REVIEW

This section is in the nature of a review. Rather than provide a broad overview of the field of quasiparticle physics, we consider several idealized problems and present their solution in detail. We then describe the bigger context which these problems were picked to illustrate, and where appropriate, the experimental situation and relevant open questions. The solutions to the problems described below will be used extensively in Section III, where we shall test a semiclassical approximation developed there on many of these cases. Although the problems considered below may at first sight seem rather different from each other, their solutions, it turns out, can all be regarded as variations on a single theme. Typically, a linearized approximation to the problem is first solved, and then the effects of the remaining ‘curvature’ terms are accounted for. In fact the same approach can be used in the study of quasiparticles in the mixed state of a d-wave superconductor, which is described in\textsuperscript{14} and references therein.

Throughout this section we will assume that the pair potential is specified, and solve for the quasiparticle spectrum in the given background. The requirement of self consistency (demanded in conventional superconductors where self-consistent mean field is a good approximation) could well change the details of the solution. However, the points we will emphasize in the following do not depend on fine details of the pair potential, and hence the simplified models we work with are able to capture these essential features. Finally, we note that for some superconductors of current interest like the cuprates that have short coherence lengths, there is no reason to believe that including self consistency into the mean field theory leads to a better approximation.

In the following we discuss some generalities relating to unconventional pairing states. The rest of the section is organised into two parts, the first deals with spin singlet superconductors and the second with spin triplet superconductors. At the beginning of each part, the appropriate Bogoliubov-deGennes (BdG) wave-equation that govern the quasiparticles is derived. The procedure for obtaining from its solutions, the excitation spectrum of the many body problem is then discussed. This needs careful attention especially in the presence of zero energy states. Next, the BdG equations are solved in several different situations. For the spin-singlet case, we first consider Andreev reflection and then derive the quasiparticle states in a region of normal metal sandwiched between two superconductors (SNS junction) and in the vortex core of an s-wave superconductor. For the spin triplet case, we consider the zero energy state at the end of a wire of superconductor. Amusingly, this explains the physics of the transverse field Ising model in one dimension. We then consider \( p_x + ip_y \) chiral superconductors, and derive the chiral edge mode and the energy zero state.
in the vortex cores of such systems and review the surprising consequences they lead to.

### A. Unconventional Pairing

The superconducting state results when electrons come together to form Cooper pairs and condense into a single state. Clearly the electron pairs can either be in a net spin zero or spin one state. In the following we consider these two cases, the spin singlet and the spin triplet superconductors, separately. In addition to the spin structure, the angular structure of the internal wavefunction of the Cooper pairs plays an important role. For example, it determines the variation of the quasiparticle gap at different points on the Fermi surface. For a rotationally invariant system, such as realized by superfluid Hes, these different pairing states can be labelled by their orbital angular momentum. Fermi statistics requires that spin-singlet (spin-triplet) superconductors be in an even (odd) orbital angular momentum state, denoted as sd, g etc.(p, f etc.). For the case of solids, only a discrete subset of the rotational symmetry is preserved due to the presence of the crystal lattice. A pairing state that spontaneously breaks an underlying symmetry of the problem (in addition to gauge symmetry) is termed an unconventional pairing state. Thus, the spin triplet p wave pairing states of superfluid Hes breaks orbital and spin rotation symmetries, and was the first example of unconventional superconductivity to be experimentally discovered\(^{15,16}\). Sometimes, even if no additional symmetry is broken, it is appropriate to use the term unconventional superconductivity if the state is fundamentally different from that of conventional superconductors. For example, in some cuprate superconductors with the copper atoms in an orthorhombic structure, the d-wave pairing does not break any lattice symmetry, in contrast to when the tetragonal structure is present. However, this state can support quasiparticle excitations with arbitrarily low energies (Dirac quasiparticles) and hence differs fundamentally from conventional superconductors. Later, we shall also discuss chiral superconductors that are characterised by a topological invariant leading to a quantized (nonzero) spin/thermal hall conductivity at zero temperature\(^{10,11}\). Again, these pairing states are clearly distinct from that of a conventional superconductor, given that they break time reversal symmetry, and are further distinguished from each other by the value of the topological invariant.

We now briefly discuss the model Hamiltonian for unconventional pairing superconductors. We specialize to the homogenous case where momentum is a good quantum number, and consider the pairing of electron states with opposite momentum \((p, -p)\). Since the pair can either be in the singlet state or in one of the three triplet states, there are four amplitudes in all needed to specify the pairing state at this momentum. The pair potential \(\Delta_p\) which is related to these amplitudes is a 2x2 matrix in spin space. Assuming that such a pair potential is given, the effective Hamiltonian for electrons then is:

\[
\mathcal{H} = \sum_p \mathcal{H}_p
\]

\[
\mathcal{H}_p = \sum_{\sigma = \uparrow, \downarrow} E(p)c^\dagger_{p\sigma} c_{p\sigma} + \frac{1}{2} \sum_{\sigma, \sigma' = \uparrow, \downarrow}\left[\Delta_p^{\uparrow \downarrow} c^\dagger_{p\sigma} \tau c^\dagger_{-p\sigma'} + \Delta_p^{\downarrow \uparrow} c^\dagger_{-p\sigma} \tau c^\dagger_{p\sigma'}\right]
\]

where \(E(p)\) the kinetic energy term, \(\tau\) is the unit anti-symmetric matrix in spin space:

\[
\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

and \([\tau c_{-p}]_{\sigma} = \sum_{\sigma'} \tau_{\sigma \sigma'} c_{-p\sigma'}\). We have chosen to write the pairing term so that the \(\tau\) matrix, which is used in defining the time reversed state of a spin half particle, appears explicitly. The advantage of writing the pairing term this way is that the singlet pair amplitude is associated with the identity matrix.

Often the following matrix notation is used in expressing the Hamiltonian:

\[
\mathcal{H} = \frac{1}{2} \sum_p \left( c^\dagger_{p\sigma} \tau c_{p\sigma} \right) \left[ E(p) \begin{pmatrix} \Delta_p & \Delta^*_{-p} \\ -\Delta^*_{-p} & -E(p) \end{pmatrix} \right] \left( c^\dagger_{p\sigma} \tau c_{-p\sigma} \right) + \text{const.}
\]

where the spin index has been suppressed, so \([c_{p\sigma}] = c_{p\sigma}\). It is easily verified that Fermi statistics for the electrons requires:

\[
\tau \Delta^T_{-p} \tau = -\Delta_p
\]

As with any general 2x2 matrix we can write \(\Delta_p\) in the form

\[
\Delta_p = \Delta_0^p \mathbf{1} + \Delta_p \cdot \mathbf{\sigma}
\]

. From (3) we require:

\[
\Delta_0^p = \Delta_0^0
\]

\[
\Delta_p = -\Delta_{-p}
\]

where we have used the identities \(-\mathbf{\tau}^2 = \mathbf{1}\) and \(\tau \sigma_i \mathbf{T} \tau = \sigma_i\). Clearly, \(\Delta_0^0\) represents spin-singlet pairing while \(\Delta_p\) (often called \(d(p)\) in the literature) represents the triplet pairing part. When \(\Delta_p\) is proportional to a vector with real components, it represents the axis in spin space along which the spin projection of the triplet pair is zero.

In conventional s-wave pairing, the singlet pair potential is taken as constant over the Fermi surface \((\Delta_0 = \Delta_0, \Delta_p = 0)\). In ‘d-wave’ superconductors, the pair potential varies over the Fermi surface and vanishes along certain directions, for example the d-wave pair potential \(\Delta_p^0 = \Delta_0^0 (p_{x}^2 - p_{y}^2) / p_{x}^2\) vanishes along a pair of
axes. An example of a triplet state is the A-phase of superfluid He$_3$, which is thought to be in the Anderson-Brinkmann-Morel pairing state given, for example, by the pair potential $\Delta_\alpha = \Delta^0 \varepsilon(p_\alpha + i p_\beta)/p_F$. We now consider the quasiparticle states in superconductors with different pairing symmetries, when translation invariance is no longer present. We begin with the case of spin singlet superconductors.

B. Spin Singlet Superconductors

1. Bogoliubov-deGennes Equation

Consider the case of a superconductor where the spin of the quasiparticle is conserved. This requires that the Cooper pairs in the condensate are in a spin singlet state, which is true both for the case of conventional superconductors as well as the d-wave pairing state seen in some high-temperature cuprate superconductors. A model Hamiltonian that describes such a spin singlet superconductor can be written in an explicitly $SU(2)_{\text{spin}}$ rotationally invariant form:

$$\hat{H} = \sum_{\sigma,\sigma'} \int d^3r \epsilon(r,r') \delta_{\sigma\sigma'} c_\sigma^\dagger r c_{\sigma'}(r') + \frac{1}{2} \Delta(r,r') \tau_{\sigma\sigma'} c_\sigma^\dagger r c_{\sigma'}(r') + \frac{1}{2} \Delta^*(r,r') \tau_{\sigma\sigma'} c_{\sigma'} r c_{\sigma}(r')$$

(5)

where $c_\sigma^\dagger r$ is the operator that creates an electron at point $r$ with spin projection $\sigma$ and $\tau$ is the unit antisymmetric matrix (2) and, from Fermi statistics we require the pairing function to be symmetric, that is $\Delta(r,r') = \Delta(r',r)$. The kinetic energy term is hermitian, $\epsilon(r,r') = \epsilon^*(r',r)$, and will often be taken as a quadratic function of momentum: $\epsilon(r,r') = \left( -\frac{s^2}{2m} - E_F \right) \delta(r-r')$.

It is convenient for the derivation below to choose a specific spin axis (call it $z$) and to then define the operators:

$$d_1^\dagger r = c_z^\dagger r$$
$$d_2^\dagger r = c_y^\dagger r$$

(6)

In terms of these ‘$d$‘ particles the Hamiltonian has no anomalous terms and reads:

$$\hat{H} = \int_{r,r'} \left( d^\dagger_1 r d^\dagger_2 r \right) \left[ \epsilon(r,r') \Delta(r,r') \Delta^*(r,r') \right] \left( d_1(r') d_2(r') \right) + \text{const.}$$

(7)

Notice, that since both the $d_1^\dagger$ and $d_2^\dagger$ create spin up particles, the density of ‘$d$‘ particles is actually the physical spin density along the $z$ axis. The absence of anomalous terms in the Hamiltonian above is just a reflection of spin conservation along this axis. Indeed, such a Hamiltonian with no anomalous terms can be written even when the $SU(2)_{\text{spin}}$ rotation invariance is broken down to $U(1)$ (spin rotations about a single axis) invariance. Hence, a Zeeman field along the $z$ direction can be accommodated above, the uniform part of the field behaves like a chemical potential for the quasiparticles. As we will see later, for spin triplet states (such as the $S_z = 0$ pairing state) where the condensate is invariant under spin rotations about a particular axis, a similar treatment applies.

The second quantized Hamiltonian (7) is quadratic and conserves the ‘$d$‘ particle number. Hence, it can be related to an equivalent wave equation called the Bogoliubov-deGennes (BdG) equation. Thus, if we solve

$$\int_{r'} \mathcal{H}_{BdG}(r,r') \psi_\alpha(r') = E_\alpha \psi_\alpha(r)$$

(8)

with the Bogoliubov-deGennes Hamiltonian :

$$\mathcal{H}_{BdG} = \left( \begin{array}{cc} \epsilon(r,r') & \Delta(r,r') \\ \Delta^*(r,r') & -\epsilon^*(r,r') \end{array} \right)$$

(9)

then the wavefunctions $\psi_\alpha(r) = (u_\alpha(r) v_\alpha(r))^T$ immediately give us the quasiparticle operators:

$$\gamma^\dagger_\alpha = \int (u_\alpha(r) d_1^\dagger r + v_\alpha(r) d_2^\dagger r) dr$$

(10)

which then diagonalize the Hamiltonian (7):

$$\hat{H} = \sum_\alpha E_\alpha \gamma^\dagger_\alpha \gamma_\alpha$$

(11)

Thus, solving the BdG equations is equivalent to diagonalizing the original second quantized Hamiltonian. From equation (10) we see that the two components of the BdG wavefunction $(u,v)$ are the amplitudes for finding the quasiparticle in the electron or hole state respectively.

One way of interpreting this procedure is as follows. The solution to the BdG equation has both positive and negative energy solutions. In the spirit of Dirac, we assume that all the negative energy levels are filled and all the positive energy levels are empty, in the ground state. Now, an excitation is made either by creating a particle in an empty level, or by destroying a particle in a filled level, both of which cost positive energy.

The BdG Hamiltonian (9) enjoys a ‘particle-hole’ symmetry. If $\psi$ is a solution to (8) with energy $E$, then $\phi = \tau \psi^*$ is also a solution but with energy $-E$. Here, $\tau$ is the unit antisymmetric matrix as defined in (2), but now acts on the two components of the wavefunction. This particle-hole symmetry of the BdG Hamiltonian implies that the excitation spectrum, derived in the manner of the previous paragraph, will be doubly degenerate. This degeneracy is simply a reflection of the $SU(2)_{\text{spin}}$ symmetry of the underlying Hamiltonian.
Finally, we note that if there exists $\psi$, a zero energy solution to the BdG equation (9), then there is a partner state $\tau\psi^*$ which is also at zero energy, but is orthogonal to $\psi$ from the antisymmetry of $\tau$. Typically, such a pair would mix and split - giving rise to a pair of $E > 0$ excitations, unless they are protected due to topology or for symmetry reasons.

We now turn to solving the BdG equations for such spin-singlet superconductors in different situations.

2. Andreev Reflection

$$\Delta$$

![Diagram: Andreev reflection: normal-superconductor interface in 1D.](image)

Consider the interface of a normal metal and a superconductor in one dimension. The pair potential $\Delta(x)$ for this situation is taken as shown in figure 1, with the metal on the left $(\Delta(x \to -\infty) = 0)$ and the superconductor on the right $(\Delta(x \to +\infty) = \Delta_0)$. The BdG equation for this situation then is:

$$\mathcal{H}_{BdG}\psi = E\psi$$

$$\mathcal{H}_{BdG} = \left(\frac{1}{2m}p^2 - E_F\right)\sigma_z + \Delta(x)\sigma_x$$

(12)

(13)

Where $\sigma$s are the Pauli matrices, but which act on the two component BdG wavefunctions. In a conventional superconductor the coherence length $\xi$ is much larger than the Fermi wavelength $(\xi k_F \gg 1)$ and so all variations of the pair potential are expected to occur over length scales large compared to $k_F^{-1}$. Therefore, if we are interested in the low energy quasiparticle states, we only need to look at states near the two Fermi points ($\pm k_F$), and scattering between them can be neglected to a first approximation (since the pair potential variation is slow compared to the Fermi wavelength). This then is the independent-node linearized approximation (or Andreev approximation$^{17}$) where we write:

$$\psi(x) \approx e^{ik_F x}\psi_R(x) + e^{-ik_F x}\psi_L(x)$$

(14)

the wavefunction $\psi_R$ and $\psi_L$ are taken to be slowly varying on the scale of $k_F^{-1}$. Substituting (14) in (13), we obtain the pair of equations:

$$(\mathcal{H}_R + \Delta\mathcal{H})\psi_R = E\psi_R$$

$$(\mathcal{H}_L + \Delta\mathcal{H})\psi_L = E\psi_L$$

where the linearized Hamiltonians are given by:

$$\mathcal{H}_R = v_F[-i\hbar\partial_x]\sigma_z + \Delta(x)\sigma_x$$

$$\mathcal{H}_L = -v_F[-i\hbar\partial_x]\sigma_z + \Delta(x)\sigma_x$$

(15)

(16)

and $v_F = p_F/m$ is the Fermi velocity. The remaining term $\Delta\mathcal{H}$ arises from the quadratic nature of the electron dispersion:

$$\Delta\mathcal{H} = -\frac{\hbar^2}{2m}\partial_x^2\sigma_x$$

(17)

and is smaller than the linearized Hamiltonians, when acting on the slowly varying functions $\psi_R$ and $\psi_L$. To begin with, it will be neglected and we just solve the linearized Hamiltonians - say $\mathcal{H}_R$ for definiteness. To illustrate the process of Andreev reflection, it is convenient to consider the zero energy eigenstate, that is, we begin in the metal with an electron at the right Fermi point ($\vec{p} = p_F \hat{x}$) and ask what happens in the presence of the superconductor. Thus we need to solve $\mathcal{H}_R\psi_R = 0$ which can be written as:

$$\partial_x\psi_R(x) = \frac{1}{v_F\hbar}\int_0^x \Delta(x') \sigma_x\psi_R(x') dx'$$

Integrating this yields:

$$\psi_R(x) = e^{\int_0^x \Delta(x') dx'} \sigma_y \psi_R(0)$$

Since all the quasiparticle excitations in the superconductor are gapped, we expect the above solution to decay as $x \to \infty$. Hence we are forced to choose the solution to be proportional to the spinor eigenstate of $\sigma_y$ with eigenvalue $-1$.

$$\psi_R(0) \propto \Phi^- = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -i \end{array} \right)$$

$$\psi \propto e^{ik_F x}\psi_R(x) = e^{ik_F x}e^{-i\frac{\pi}{4}}\int_0^x \Delta(x') dx' \left( \begin{array}{c} 1 \\ -i \end{array} \right)$$

Deep within the superconductor, this solution decays as $e^{-i\frac{\Delta}{k_F}x}$ with a length scale equal to the coherence length $\xi = \hbar v_F/\Delta$. In the metal ($x \to -\infty$) however, the unnormalized wavefunction is:

$$\psi(x \to -\infty) = \left( \begin{array}{c} e^{ik_F x} \\ -ie^{ik_F x} \end{array} \right)$$

Thus the incident electron (upper component) with momentum $p_F$ is reflected by the superconductor into a hole with the same momentum and hence opposite velocity. The difference in electrical charge is taken up by the superconductor that absorbs a Cooper pair into the condensate. This is the process of Andreev reflection; as pointed out in the original paper$^{17}$ it leads to a reflection of the quasiparticle and heat currents, but not of the electrical current. For quasiparticle states at $E \neq 0$ (away from zero energy) an electron with momentum $p_F + E/v_F$ is reflected as a hole with momentum $p_F - E/v_F$. 

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3. SNS Junction

Now consider the case where a piece of normal metal (N) is sandwiched between two superconductors (S) to give a SNS junction. For quasiparticle energies less than the superconductor gap, bound states can appear in the region between the two superconductors. We will be interested in solving the BdG equations for these bound states.

The simplest model of a SNS junction has a pair potential profile as shown in figure 2, where it is taken to have a step profile with equal magnitudes in the two superconducting regions, but a relative phase difference $\phi$ between the superconductors:

$$\Delta(x) = \begin{cases} 
\Delta_0 e^{i\phi} & x < -L \\
0 & -L \leq x < 0 \\
\Delta_0 & 0 \leq x
\end{cases}$$

FIG. 2. A Superconductor-Normal-Superconductor junction with relative phase difference $\phi$. For simplicity, the pair potentials are assumed to have the step profiles shown.

We can now solve the BdG equations in these three regions and match up the solutions. We begin with the linearized approximation derived previously, and although the step profile considered here does not qualify as a smooth pair potential for which this approximation is strictly valid, we will see at the end that the salient features of the solution remain when a smooth pair potential is considered.

In the right superconductor ($x > 0$) the BdG equations for the quasiparticles near the (say) $+p_F$ Fermi point is:

$$[\psi_F(-i\hbar \partial_x)\sigma_z + \Delta_0 \sigma_x] \psi_R(x) = E \psi_R(x)$$

which can be integrated to give the solution:

$$\psi_R(x) \propto e^{-\frac{\Delta(x)}{2\hbar} \sqrt{\frac{\Delta_0 - E^2}{\Delta_0}}} e^{-i\frac{\Delta_0 x}{2\hbar}} \left( \frac{1}{-i} \right)$$

which decays as $x \to -\infty$, where we denote:

$$e^{i\theta_F} = \sqrt{1 - \frac{E^2}{\Delta_0^2} + i \frac{E}{\Delta_0}}$$

Similarly, for the superconductor on the left ($x < -L$), the appropriate solution obtained by integration

$$\psi_R(x) \propto e^{i\theta_F + \pi - \phi} \sigma_z e^{\frac{\Delta_0}{2\hbar} \sqrt{\frac{\Delta_0^2 - E^2}{\Delta_0}}} \left( \frac{1}{-i} \right)$$

decays as $x \to -\infty$. The free electron solution in the normal region gives us the matching condition:

$$\psi_R(0) \propto e^{iE \frac{\Delta_0}{2\hbar} \sigma_z} \psi_R(-L)$$

Putting the three (18,19,20) together, we obtain the quantization condition for the energy $E$:

$$\theta_E + \frac{E}{\hbar \psi_F} \frac{L}{\Delta_0} = \frac{1}{2} (\phi - \pi) + \pi n$$

where $n$ is an integer. Once the eigenvalues for the $+p_F$ Fermi point is solved, the solution to the $-p_F$ Fermi point is obtained by recognizing that if

$$\mathcal{H}_R \psi_R = E \psi_R$$

$$\Rightarrow \mathcal{H}_R (\tau \psi_R^*) = -E (\tau \psi_R^*)$$

where $\tau$ is the unit antisymmetric matrix.

There are several interesting features associated with this spectrum, especially its evolution as the relative phase $\phi$ is changed. As discussed nicely in $^{18}$, this plays an important role in understanding the Josephson effect in such systems. However, the feature that we would like to emphasize is that at $\phi = \pi$, there is a zero energy solution to (21). That is, within the linearized approximation there is a zero energy state in the $\pi$ shifted SNS junction. The unnormalized wave function of this zero energy state is:

$$\psi_R(x) = f(x) \left( \begin{array}{c}
1 \\
-i
\end{array} \right)$$

where

$$f(x) = \begin{cases} 
e^{-\Delta_0 x / \hbar \psi_F} & x > 0 \\
1 & -L \leq x < 0 \\
ne^{-\Delta_0 (x + L) / \hbar \psi_F} & x < -L
\end{cases}$$

Besides, by the relation (22) there is a partner state at zero energy. This pair of zero energy states is not particular to the step profile that we have chosen. Rather, it exists quite generally as long as the order parameter changes sign and is real everywhere. For a general pair potential with these properties, the unnormalized wave function of the zero energy states has the form (23) but with the function $f(x) = e^{-\Delta_0 x / \hbar \psi_F} \int_0^x \Delta(x') dx'$ which decays as $x \to \pm \infty$ if $\Delta(x \to \pm \infty) > 0$ and $\Delta(x \to -\infty) < 0$.

A very similar situation occurs in d-wave superconductors at surfaces with a particular orientation. Quasiparticles reflected from the surface see a pair potential that effectively changes sign, due to the momentum dependence of the pair potential. This leads to low energy
bound states at the surface. These states are believed to have been observed in tunneling experiments that see an enhanced density of states near zero energy\textsuperscript{19,20}.

One question that is of interest is whether these states of the $\tau$ shifted SNS junction remain at zero energy on going beyond the linearized approximation. It turns out that the curvature term $\Delta \mathcal{H}$ (17) mixes the two solutions and gives rise to a small splitting. This splitting can be easily calculated by projecting the full Hamiltonian into the two dimensional subspace generated by this pair of states. For a smooth potential, the splitting is rather small $\Delta E \sim \frac{\Delta}{E_F}$.

4. Vortex Bound States: $s$-wave pairing

Consider a vortex in an $s$-wave superconductor. At the vortex center the pair potential is reduced to zero, hence there can exist low energy quasiparticle states that are localized in this region. These vortex bound states were first predicted many years ago in\textsuperscript{3}. Although this classic calculation is well known, we revisit this problem using an alternate approach more in line with what we have used in the previous sections. That is, we begin by looking at a linearized version of the problem that can be solved easily, and then include the effects of the curvature terms as a perturbation.

For simplicity imagine an isolated vortex with rotational symmetry, with a pair potential magnitude that is strictly zero up to some length scale $L$, and then steps up to its bulk value $\Delta_0$:

$$\Delta(r) = \begin{cases} 0 & r < L \\ \Delta_0 & r > L \end{cases}$$

Usually, the size of the vortex core $L$ is assumed to be of order the coherence length $\xi$, although at low temperatures, theoretical and numerical work indicate that the core size shrinks\textsuperscript{21,22}. In the following we consider an isolated vortex and ignore for now motion along the vortex line. For an extreme type II superconductor, the effect of the magnetic field can be neglected. This is because the region in which these low energy states are localised is of area $\xi^2$, which barely encloses any of the magnetic flux that is spread a much larger area controlled by the penetration depth ($\lambda_L$). Then, the BdG Hamiltonian to solve is:

$$\mathcal{H} = \left( \frac{\mathbf{p}^2}{2m} - E_F \right) \mathbf{\sigma} + \Delta(r) e^{-i\theta} \mathbf{\sigma}^+ + \Delta(r) e^{i\theta} \mathbf{\sigma}^-$$

where $r$ and $\theta$ are the radial and angular coordinates. Due to the circular symmetry of the vortices, the eigenstates can be labelled by a quantum number $\mu$ that is like the angular momentum, and can be written as:

$$\psi_\mu(r, \theta) = \frac{e^{i\mu \theta}}{\sqrt{r}} \begin{pmatrix} e^{-i\mu \theta} f_1(r) \\ e^{i\mu \theta} f_2(r) \end{pmatrix}$$

where by single valuedness we require that $\mu$ be a half integer $\mu = m + \frac{1}{2}$, $m \in \mathbb{Z}$. The eigenvalue equation now takes the form:

$$\mathcal{H}_\mu \begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} = E \begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix}$$

$$\mathcal{H}_\mu = \frac{\hbar^2}{2m} \left[ (-\partial_r^2 - k_F^2 + \frac{\mu}{r^2}) \sigma_z - \frac{\mu}{r^2} 1 \right] + \Delta_0 \sigma_x$$  (26)

We first solve this equation in the core region, then outside, and finally match these two solutions. In the core region $r < L$, since the pair potential vanishes, the eigenfunctions are easily seen to be Bessel functions. The upper and lower components of the quasiparticle wavefunction are independent of each other in this region:

$$f_1(r) \propto \sqrt{r} J_{\mu - \frac{1}{2}} \left( |k_F + \frac{E}{v_F}| r \right)$$

$$f_2(r) \propto \sqrt{r} J_{\mu + \frac{1}{2}} \left( |k_F - \frac{E}{v_F}| r \right)$$  (27)

Outside the core region ($r > L$), we have the effect of the pair potential to take into account. Luckily, a brief inspection shows that we can make the linearized approximation in this region. If we denote the spinor $f(r) = (f_1(r), f_2(r))^T$, we can write the dependence on the radial coordinate as follows:

$$f(r) = e^{ik_F r} f_+(r) + e^{-ik_F r} f_-(r)$$  (28)

Then, neglecting scattering with large momentum ($2p_F$) transfer we can write the eigenvalue equation as:

$$(\mathcal{H}_+ + \Delta \mathcal{H}) f_+ = E f_+$$

$$(\mathcal{H}_- + \Delta \mathcal{H}) f_- = E f_-$$  (29)

where the linearized piece of the Hamiltonian takes on the simple form:

$$\mathcal{H}_+ = -i v_F \partial_r \sigma_x + \Delta_0 \sigma_x$$

$$\mathcal{H}_- = i v_F \partial_r \sigma_x + \Delta_0 \sigma_x$$

and the curvature term is given by:

$$\Delta \mathcal{H} = -\frac{\hbar^2}{2m} \frac{1}{r^2} + \frac{\hbar^2}{2m} \left[ \frac{\mu^2}{r^2} - \partial_r^2 \right] \sigma_x$$  (30)

As usual, we first solve the linearized part of the Hamiltonian, and subsequently include the effects of the smaller terms $\Delta \mathcal{H}$, that arise from the curvature of the electron dispersion (and hence depend explicitly on the electron mass). The solutions to these linearized equations that decay at $r \to \infty$ have already been obtained in the previous section, in the context of the bound states in SNS junctions (18,22). We now match these solutions against the ones in the vortex core. For core sizes that are much larger than the Fermi wavelength, the asymptotic form of the Bessel functions can be used ($k_F < r < L$):
where $\epsilon = \frac{E}{v_F \hbar}$. The incoming and outgoing waves are matched separately at $r = L$ which give rise to the quantization condition on the energies:

$$E = \pm \Delta_0 \sin(2EL/\hbar v_F) \quad (31)$$

The only low energy ($E \ll \Delta_0$) solution of this equation is $E = 0$. Thus, within the linearized approximation there is a branch of quasiparticle states $\{|\mu >\}$, labelled by $\mu$, that are all at zero energy. The curvature terms lift this degeneracy. Their effect may be taken into account within first order perturbation theory, where the shift in the energy levels they lead to can be written as:

$$\delta E_\mu = \frac{\langle \mu | \Delta H | \mu >}{\langle \mu | \mu >} \quad (32)$$

It is clear that the last two terms of $\Delta H$ which contain $\sigma_z$ do not contribute to (32) since the quasiparticle spinor that is the solution of the linearized equations is along the $\hat{y}$ direction. Thus, the remain term yields:

$$\delta E_\mu = -\mu \frac{\hbar^2}{2m} \frac{1}{\langle \mu | \mu >} \int_L^\infty \frac{dr}{r^2} e^{-2r/\xi} \quad (33)$$

where $\langle \mu | \mu >$ denotes the normalization of our wavefunctions. For a core size $L \sim \xi$, we find

$$\delta E_\mu \approx -\mu \frac{\Delta_0^2}{E_F} \quad (34)$$

Thus, there is a low energy branch of quasiparticle states, bound to the vortex core. On adding the motion along the vortex line, these states are broadened into one dimensional bands, the states of which are labelled by the quasiparticle wavenumber $k$, along the vortex line. STM measurements in the vortex state of the conventional superconductor NbSe$_2$, have provided direct evidence for the existence of these vortex quasiparticle states. Very surprisingly, STM measurements on the cuprate superconductors also seem to reveal a (fairly high energy) bound state in the vortex core. This is still a mystery, since a d-wave superconductor is not expected to possess any vortex bound states. We return briefly to this puzzle in the last section of the article.

C. Spin Triplet Superconductors

1. Bogoliubov deGennes Equation

Here we derive the BdG equations for spin triplet pairing superconductors. A discussion of the wide variety of spin triplet states can be found elsewhere, here we focus on two particular pairing situations, that will be repeatedly referred to in the following, and that suffice to bring out some of the features of the triplet pairing superconductors. The two triplet states we will discuss are: the pairing state with spin projection zero along a fixed direction (called the $S_z = 0$ state in the following) and the fully spin polarized triplet state. In the notation of Section II A, the $S_z = 0$ state is represented by $\Delta_p = \Delta(p) \vec{z}$ while the spin polarized state takes the form $\Delta_p = \Delta(p)(\hat{x} \pm i\hat{y})$.

The $S_z = 0$ state has spin rotation invariance about the $z$ axis, and hence the $z$ component of the quasiparticle spin is conserved. This allows us to use the ‘d’ particle representation of Section II B to obtain a Hamiltonian with no anomalous terms. In the spin basis along $z$ the model Hamiltonian takes the form:

$$\hat{H} = \int_{r,r'} \sum_{\sigma = \uparrow, \downarrow} \varepsilon_{\sigma}(r,r') c_{\sigma}^\dagger(r') c_{\sigma}(r')$$

$$+ \frac{1}{2} \Delta(r,r') c_{\uparrow}^\dagger(r') c_{\downarrow}(r')$$

$$- \frac{1}{2} \Delta^*(r,r') c_{\downarrow}^\dagger(r') c_{\uparrow}(r') \quad (35)$$

where $\varepsilon(r,r') = \varepsilon^*(r',r)$, and from Fermi statistics for the spin triplet state the pairing function is odd under interchange of space indices: $\Delta(r,r') = -\Delta(r',r)$. Introducing the ‘d’ particles as before (6) we obtain:

$$\hat{H} = \int_{r,r'} \left( d_{1}^\dagger(r') \ H_{BdG}(r',r') \ d_{1}(r') \right) + \text{const.}$$

$$\ H_{BdG} = \left( \begin{array}{cc} \varepsilon(r,r') & \Delta(r,r') \\ -\Delta^*(r,r') & -\varepsilon^*(r,r') \end{array} \right) \quad (36)$$

Note that the difference from the singlet case (7) arises due to the odd parity of $\Delta(r,r')$ that yields the negative sign before the $\Delta^*$ term. Since we have a number conserving quadratic Hamiltonian, there is an equivalent wave equation - the BdG equation that reads:

$$\int_{r'} \ H_{BdG}(r,r') \psi_{\alpha}(r') = E_{\alpha} \psi_{\alpha}(r) \quad (37)$$

with the BdG Hamiltonian as defined in (36). Again, once the BdG equation is solved for the energy levels, all the negative energy levels are assumed to be filled. The excitation spectrum of the many body problem is then obtained by creating particles in the positive energy unoccupied levels, or destroying particles in the filled negative energy levels, both of which lead to positive energy excitations. The expression for the quasiparticle creation operator and many body Hamiltonian in terms of the solution to the BdG equations are the same as in the spin singlet case (10,11).

The BdG Hamiltonian (36) possesses a ‘particle-hole’ symmetry. If $\psi$ is a solution to (37) with energy $E$, then

$$\phi = \sigma_z \tau \psi^* \quad (38)$$
with $\sigma_z$ in the usual representation, is also a solution but with energy $-E$. This particle hole symmetry implies that the excitation spectrum ($E > 0$) will be doubly degenerate. Here, the degeneracy arises because the $z$ component of the quasiparticle spin is a good quantum number and there is the discrete symmetry under $\hat{z} \rightarrow -\hat{z}$.

The crucial difference from the spin singlet case arises when we consider the possibility of an $E = 0$ quasiparticle state. While in the spin singlet case we found that such a solution to the BdG equations always has a linearly independent partner, at the same energy, this is not necessarily so for the spin triplet case. In fact, if $\psi$ is an energy zero solution to the BdG equations, then the state $\phi$ defined above (38) also is at zero energy but is not necessarily independent of $\psi$, and it is possible to have just a single state at zero energy. Let us examine the physical consequences if that were to happen - subsequently we will point out specific situations (in the vortex core of a 2D $p_x + i p_y$ superconductor or at the end of a 1D wire of $p_x$ superconductor) where this is actually realised. When the BdG equation has a single zero energy solution, the many body system (35) has a pair of degenerate ground states that correspond to this level being either empty or occupied. Since the $d^i$ operators both raise the spin projection along the $z$ direction by one half, the two ground states differ by $\Delta S_z = \pm \hbar$. The only symmetric way to assign $S_z$ quantum numbers to these two ground states is $\pm \frac{1}{2} \hbar$. The $z$ component of the electron spin has been fractionalised! This is very analogous to the fractionalization of the electron in one dimensional charge density wave systems\(^{12}\).

**Spin Polarised Case:** The other triplet pairing state that we consider in some detail is the pairing state of spin polarised (or equivalently, spinless) fermions. Here, rather than working in the continuum, we imagine a set of spinless fermions that live on a lattice whose sites are labelled by $i$. Then the model Hamiltonian takes the simple form:

$$\hat{H} = \sum_{i,j} \left[ c^\dagger_i t_{ij} c_j + \Delta_{ij} c^\dagger_i c^\dagger_j + \Delta_{ij}^* c_i c_j \right]$$

(39)

where $t_{ij}^* = t_{ji}$ and $\Delta_{ij} = -\Delta_{ji}$. To find the excitation spectrum of this Hamiltonian we construct quasiparticle operators ($\gamma^0_\alpha$) that create a quasiparticle in state $\alpha$ with energy $E_\alpha$. The operator then must satisfy:

$$[\hat{H}, \gamma^\dagger_\alpha] = E_\alpha \gamma^\dagger_\alpha$$

(40)

If we expand the quasiparticle operator in particle and hole creation operators:

$$\gamma^\dagger_\alpha = u^\dagger_\alpha c^\dagger_i + v^\dagger_\alpha c_i$$

(41)

(summation over repeated indices is assumed above and henceforth), then the coefficients of the expansion satisfy:

$$\begin{pmatrix} t_{ij} & \Delta_{ij} \\ -\Delta_{ij}^* & -t_{ij} \end{pmatrix} \begin{pmatrix} u^\dagger_\alpha \\ v^\dagger_\alpha \end{pmatrix} = E_\alpha \begin{pmatrix} u^\dagger_\alpha \\ v^\dagger_\alpha \end{pmatrix}$$

(42)

this is the discrete version of the BdG equation in (37). This BdG equation also possesses a particle hole symmetry; if $(u^\dagger, v^\dagger)$ has energy $E$, then, under the transformation:

$$\begin{pmatrix} u^\dagger \\ v^\dagger \end{pmatrix} \rightarrow \begin{pmatrix} v^\dagger \\ u^\dagger \end{pmatrix}$$

(43)

an energy $-E$ state is obtained. However, unlike in the earlier situations that we considered, this particle hole symmetry does not lead to a degeneracy of the excitation spectrum. This can be seen by constructing the quasiparticle operators corresponding to these two solutions of the BdG equations. If $\gamma^\dagger = u^\dagger c^\dagger_i + v^\dagger c_i$, then the quasiparticle operator corresponding to the transformed state is $v^\dagger c^\dagger_i + u^\dagger c_i$, which is just $\gamma$, and hence no new excitation results. Hence, we just solve (42) for the positive energy levels, which then yields the quasiparticle operators (41).

The state at zero energy, should one exist, is particularly interesting. Assume that there is a single state, which must therefore be taken to itself under the transformation (43). Then it is always possible to choose the overall phase so that we have:

$$u_0^\dagger = v_0^\dagger$$

and hence the quasiparticle operator that creates this zero mode $\gamma^\dagger_0 = u^\dagger c^\dagger_i + v_0^\dagger c_i$ satisfies:

$$\gamma^\dagger_0 = \gamma_0$$

(44)

That is, it is a real (Majorana) fermion. Thus, zero energy states in the BdG equation, in this case, lead to the appearance of Majorana fermions which again is a signal of fractionalization of the quasiparticles in these systems. Exotic states of this kind have been considered in the context of quantum computing, where they have been proposed as hardware for quantum bits. The fact that these states are fractionalized, and hence possess quantum numbers different from that of ordinary matter makes them stable against decoherence and errors\(^{13}\). Therefore they have been argued to be ideal as qubits of a quantum computer.

Finally we make a brief comment about triplet pairing with different spin structures. Actually, the Hamiltonian (39) represents the most general pairing Hamiltonian, if we take the index ‘i’ to run over both space and spin indices. In the presence of additional symmetries, such as $SU(2)_{spin}$ or $U(1)_{spin}$, the analysis of the sections which dealt with those particular cases is more appropriate, since they address directly the consequences of the symmetries. In the absence of such symmetry, the discussion above (which for simplicity is presented for the case of spinless fermions) applies. We now consider the BdG equation for spin triplet superconductors in various interesting situations.
2. Quasiparticle States at the End of a Wire:

We have just seen that zero energy states of a triplet superconductor lead to unusual consequences. Here we will display a simple physical situation where such zero energy states can be realized. Consider a triplet superconductor in one spatial dimension, for simplicity we assume it is of the $S_z = 0$ or of the spin polarised variety. We will see that if the superconductor is terminated, then there is a zero energy state at the end. One may well ask if this is of any relevance to experimental systems. In fact, there is growing evidence that the organic compound (TMTSF)$_2$PF$_6$ is a spin triplet superconductor$^{26-28}$, although all the details of the pairing have yet to be established. For example, NMR Knight shift measurements, which probe the spin susceptibility, shows no sign of suppression on entering the superconducting state, as would happen in a singlet superconductor$^{27}$. Also, superconductivity survives the application of large Zeeman fields, exceeding the so called Clogston-Chandrasekhar limit beyond which the Zeeman energy would destroy singlet pairing$^{28}$. Given also that these compounds are quasi-one dimensional in nature, they could well serve as an experimental system where some of the physics reviewed below is realized. In fact, tunneling into such end states has been proposed as a means to identify the pairing symmetry in these compounds$^{29}$. Utilizing such end states as qubits for quantum computation was suggested in$^{30}$. Also, as we will see later, the transverse field Ising model in one dimension is equivalent to such a triplet superconductor made of spinless fermions, and the details of that spin model can be understood very simply by the solution to the BdG equations described below.

\[ \Delta(p) = \frac{\Delta_0}{p_F} p. \]  
Consider a semi-infinite wire of such a superconductor. Since the gap is of opposite sign at the two Fermi points, we would expect that when a quasiparticle is reflected at the end it leads to a suppression in the gap. Therefore, an appropriate profile for $\Delta$, has it vanishing near the end. As usual, we choose to model this suppression by setting $\Delta = 0$ over a region of length $L$ adjacent to the end, and a sharp transition just beyond, up to the bulk value $\Delta = \Delta_0^*$, as shown in figure 3. At the end $x = 0$, the quasiparticles are assumed (say) to satisfy Dirichlet boundary conditions.

We now consider the linearized version of this problem, expanding the quasiparticle wavefunction over the excitations near the two $(R, L)$ Fermi points:

\[ \psi(x) \cong e^{ik_F x} \psi_R(x) + e^{-ik_F x} \psi_L(x) \]

then the linearized Hamiltonian at the two Fermi points are given by:

\[ \mathcal{H}_R = v_F (-i \partial_x) \sigma_z + \Delta(x) \sigma_x \]
\[ \mathcal{H}_L = -\mathcal{H}_R \]

and the smaller curvature terms:

\[ \Delta \mathcal{H} = -\frac{\hbar^2}{2m} \partial_x^2 \sigma_z + \frac{\Delta_0}{p_F} (-i \partial_x) \sigma_x \]

will be taken into account subsequently.

The linearized equations have a zero energy solution (23)

\[ \psi_{R/L}(x) = f(x) \left( \begin{array}{c} 1 \\ -i \end{array} \right) \]

with

\[ f(x) = \begin{cases} e^{-\frac{\Delta_0}{p_F} (x-L)} & x > L \\ 1 & L > x > 0 \end{cases} \]

. These can be now stitched together to obey the correct boundary conditions at $x = 0$. Thus, the single zero energy state we obtain is:

\[ \psi(x) = (e^{ik_F x} - e^{-ik_F x}) f(x) \left( \begin{array}{c} 1 \\ -i \end{array} \right) \]

Note, that just like in the case of the $\pi$ shifted SNS junction (Section II.B.3), where the quasiparticle sees a pair potential that changes sign, here too the quasiparticle that is reflected from the end of the wire has its momentum reversed, and hence sees a pair potential of opposite sign. However, there is one crucial difference

*Ordering the operators p,x correctly is an issue that is best resolved by going over to an equivalent lattice model. We will comment on subtleties as they arise.
between the two cases. While in the spin singlet case, the \( \pi \) shifted SNS junction had a pair of zero modes in the linearized approximation, here we just have a single state at zero energy. The particle hole transformation for these triplet state quasiparticles, \( \psi \rightarrow \sigma_z \tau \psi^* \) does not yield a distinct state at zero energy.

The fact that there is a single zero energy state at the end of the chain makes it stable with respect to perturbations. For example, including the effect of the curvature terms \(^1\) does not move this state away from zero energy since particle hole symmetry needs to be preserved, and there is no partner for this state, should it attempt to move away from zero energy. For a finite length of wire, there are zero modes at each end, which can then mix and give rise to a small splitting that moves them away from zero energy. However, since the wavefunction of these states decay exponentially through the bulk, the splitting is expected to drop as \( \sim e^{-l/\xi} \) where \( l \) is the length of the wire and \( \xi = \hbar v_F / \Delta_0 \). These end states have been invoked in\(^{31} \) to distinguish different phases of dirty superconductors in one dimension.

According to our general discussion on the physics of zero modes in triplet superconductors, we have two different interpretations of these modes depending on whether the BdG equations above were derived for a \( S_z = 0 \) pairing state or the pairing state of spin polarised fermions. In the first case, the zero mode at the end can be either occupied or empty, and leads to a pair of ground states of the system with \( S_z = \pm \frac{1}{2} \). For the spin polarised case, it implies that there is a Majorana fermion mode, localised at the end.

Although for superconductors of interest, we are in the weak pairing regime, in the sense that \( \mu > 0 \), it is of theoretical interest to consider the strong pairing case, when \( \mu < 0 \). There the superconductor can be viewed as a Bose condensate of tightly bound molecules made up of pairs of electrons. Interestingly, the zero modes that we obtained at the ends are no longer present in the strong pairing limit. We can follow the evolution of these zero energy states starting from the weak pairing regime, as the chemical potential is gradually reduced down to negative values. When the chemical potential passes through zero, the bulk supports gapless excitations, which release the zero mode from the edge, and allows it to mix with the mode at the far end of the wire. Thus, once we are in the strong pairing regime, the zero mode has disappeared from the ends.

\(^1\)In fact, even without linearization, the quadratic Hamiltonian can be easily solved for the zero energy state.

3. Transverse Field Ising Model and Superconductivity

The physics of the transverse field Ising model in one dimension can be understood in terms of the p-wave superconductor problem we have studied above. Consider a spin half chain of \( N \) spins with the interaction

\[
H_{\text{Ising}} = -J \sum_{i=1}^{N-1} S_i^z S_{i+1}^z + \hbar \sum_{i=1}^{N} S_i^x
\]

where we have (say) a ferromagnetic \( J > 0 \) coupling between the \( z \) components of the spins, and a disordering field \( h \) along the \( x \) direction. The operators for spin half, can be represented as Pauli matrices on each site \( \hat{S} = \frac{1}{2} \sigma \).

The physics of this model is well known which we briefly review here. In the thermodynamic limit \( N \rightarrow \infty \) this system has two phases separated by a quantum phase transition. When the ferromagnetic coupling is large, \( |J/h| > 1 \), the spins order along the \( \pm z \) direction, thereby spontaneously breaking the \( z \rightarrow -z \) symmetry present in the Hamiltonian. In this phase the ground state is doubly degenerate. As the disordering field \( h \) becomes strong, it destroys the magnetization along the \( z \) axis, leading to a phase with a unique ground state.

Let us rewrite this problem within the fermion representation of spin half chains. Introducing the spinless fermion operators \( c_i \) at each site, and choosing the quantization axis along the \( x \) direction, we can express the spin operators as:

\[
\begin{align*}
S_j^x &= c_j^\dagger c_j - \frac{1}{2} = n_j - \frac{1}{2} \\
S_j^y &= \left( \prod_{k<j} (-1)^{n_k} \right) \frac{1}{2} (c_j^\dagger - c_j) \\
S_j^z &= \left( \prod_{k<j} (-1)^{n_k} \right) \frac{1}{2} (c_j^\dagger + c_j)
\end{align*}
\]

Using the canonical anticommutation relations for fermions, it is easily shown that the spin operators as written above obey the correct spin algebra, commute with spin operators on a different site, and correspond to spin half. In terms of these fermions, the Hamiltonian then reads:

\[
\frac{1}{J} H_{\text{Ising}} = -\frac{1}{2} \sum_{i=1}^{N-1} (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + \hbar \sum_{i=1}^{N} c_i^\dagger c_i
\]

\[
-\frac{1}{2} \sum_{i=1}^{N-1} (c_i c_{i+1} - c_{i+1}^\dagger c_i^\dagger)
\]

Clearly this is the Hamiltonian for a p-wave superconductor of spinless fermions, on a one dimensional lattice, where the parameter \( \nu = h/J \) controls the chemical potential. For the parameter range \( |\nu| < 1 \), the chemical potential lies within the fermion band, and this is identified as the weak pairing regime. A straightforward analysis...
of this lattice model reveals that for a semi-infinite chain, a zero energy state exists in the weak pairing regime but not in the strong pairing regime ($|\nu| > 1$). Since we are dealing with spinless fermions, this zero energy state implies a Majorana degree of freedom associated with the end. For a finite chain with $N$ sites in the weak pairing phase, the zero modes at the two ends of the chain mix and split. Due to the presence of the bulk gap, this splitting is exponentially small $\sim |\nu|^N$. In the thermodynamic limit, both these modes are at zero energy and may be combined to give a single Dirac level that can either be occupied or empty, that leads to a two fold degeneracy of the ground state in the weak pairing (ferromagnetic) phase. As the chemical potential moves out of the band ($|\nu| = 1$), the gap collapses and the Majorana fermions are free to move in the bulk. The zero energy states then combine and disappear. Thus, in the strong pairing phase there are no zero modes and hence a unique ground state is obtained.

4. Chiral Superconductors

In this section we will be concerned with chiral superconductors, where parity and time reversal symmetry are broken by the pair potential. Examples include the $d_x^2 - d_y^2 \pm i d_{xy}$ singlet pairing state, and the $p_x \pm i p_y$ triplet pairing state. Quasiparticles in a two dimensional chiral superconductor exhibit an analog of the Quantized Hall Effect, seen in two dimensional electron systems in a strong magnetic field. However, unlike electrons, the charge of the quasiparticle is not conserved due to the presence of the superconductor condensate. Hence what is quantized is the thermal Hall or spin Hall (if quasiparticle spin is conserved) conductivity. A two dimensional chiral pairing state can be realized in thin films of superfluid He$_3$ in the A or A$_1$ phase. Then, the nodal direction is expected to orient itself normal to the plane, effectively yielding a $p_x \pm i p_y$ pairing state. Another candidate chiral superconductor is the layered compound Sr$_2$RuO$_4$. Early work predicted a spin triplet superconductor, and subsequent experiments, especially muon spin resonance experiments that showed signage of a time reversal symmetry broken state led to the proposal that these were $p_x \pm i p_y$ superconductors. However, recent experiments have cast doubt on this identification, and the pairing state remains to be definitively established. In the context of the fractional quantum Hall effect, the paired states proposed to account for the experimentally observed fraction at $\nu = 5/2$, can be thought of as chiral pairing states of composite fermions. For example, the Read-Moore state consists of spin polarised composite fermions paired in a $p_x \pm i p_y$ state. This proposal has been receiving increasing support from experiments and numerical calculations, as the most likely candidate for the $\nu = 5/2$ fraction. The specific system we shall consider in the following is the $p_x + i p_y$ superconductor (with spin structure: $S_z = 0$ or spin polarised). First, we show the existence of chiral modes at the edge. Next, we consider vortices in such a superconductor, and find that they possess zero energy bound states.

(i) Chiral Edge Modes in a $p_x + i p_y$ Superconductor

The BdG equations corresponding to such a $p_x + i p_y$ superconductor is:

$$\mathcal{H} = \left( \frac{p^2}{2m} - \mu \right) \sigma_z + \frac{\Delta_1}{p_F} p_x \sigma_x + \frac{\Delta_2}{p_F} p_y \sigma_y$$

where we have set $\Delta_1$ and $\Delta_2$ to be the strengths of the $p_x$ and $p_y$ components respectively. Now consider an edge of the system along the $y$ direction ($x = 0$), with the superconductor on the right. As in the previous example we assume appropriate boundary conditions on the quasiparticle wavefunction such as; $\psi(x = 0, y) = 0$. Also, as argued previously, reflection will suppress the pair potential $\Delta_1$, hence the model pair potential we work with is

$$\Delta_1(x) = \begin{cases} 
0 & 0 < x < L \\
\Delta_0 & L < x 
\end{cases}$$

The pair potential $\Delta_2$ however is taken to be constant since reflection does not change the momentum along the $y$ direction. Using the translation invariance along the edge, we can write $\psi(x, y) = e^{i k_y y} \psi_{k_y}(x)$, and the BdG Hamiltonian then takes the form:

$$\mathcal{H}_{k_y} = \left[ \frac{\hbar^2}{2m} (-\partial_x^2 - k_y^2) \sigma_z + \frac{\Delta_1}{k_F} (-i \partial_x) \sigma_x \right] + \frac{\Delta_2}{k_F} k_y \sigma_y$$

where $k_y^2 = k_x^2 - k_y^2$. Notice, that the part of the Hamiltonian within square brackets has been solved in the section before, for the zero energy state at the end of a wire. The zero energy state obtained there (in the linearized approximation) that satisfies the boundary condition is:

$$\psi_{k_y}^{(0)} = f(x) \sin(k_F x) \left( \frac{1}{i} \right)$$

where the function $f(x)$ is the same as in that case \ref{eq:49}, but with the modified Fermi velocity $v_F = \frac{1}{m} \sqrt{k_x^2 - k_y^2}$.

Amusingly, this is an eigenstate of the remaining part of the Hamiltonian \ref{eq:55} as well, since it is an eigenstate of $\sigma_y$. Thus:

$$\mathcal{H}_{k_y} \psi_{k_y}^{(0)} = -\frac{\Delta_2}{k_F} k_y \psi_{k_y}^{(0)}$$

Thus, we have obtained a low energy branch of eigenstates with dispersion:

$$E(k_y) = -\frac{\Delta_2}{k_F} k_y$$
which represents a mode moving down the edge with velocity $\Delta_2/p_F$. This is the chiral mode that arises from the non-trivial topological properties of chiral superconductors. It can now be argued from the chiral nature of this mode, that perturbations such as the curvature terms or changes in the gap profile might change its detailed properties, but would not destroy it. The only relevant perturbation would be tunneling of quasiparticles to an edge propagating in the opposite direction.

For the case of the $S_z = 0$ pairing state, this implies the existence of a single chiral Dirac mode at the edge. Given that these are the only low energy fermion excitations of the system, simple arguments\(^{38}\) reveal that the thermal Hall conductance will be quantized, that is the ratio of the thermal Hall coefficient $\kappa_{xy}$ to the temperature $T$ is a universal constant: $\frac{\kappa_{xy}}{T} = \pm \frac{\pi^2 k_B^2}{3h}$. (This is the same as the thermal Hall transport of a single chiral mode of electrons in the usual quantum Hall effect, that would occur, say, at $\nu = 1$). Since the $z$ component of quasiparticle spin is also conserved, there is quantized spin hall transport of that component of spin as well. For the case of pairing of spin polarized fermions, only half the degrees of freedom counted above are physical, and hence we obtain a chiral Majorana mode at the edge from this BdG solution. These transport only half as much heat as the Dirac particles and hence the thermal Hall transport is quantized to $\frac{\kappa_{xy}}{T} = \pm \frac{\pi^2 k_B^2}{6h}$.

(ii) Vortex Zero Mode in the $p_x + i p_y$ Superconductor

In this section we consider the BdG equations for a single vortex in a layer of $p_x + i p_y$ superconductor\(^{39}\). While this problem may appear very similar to that of a vortex in an s-wave superconductor, discussed in section II B 4, there is a crucial difference arising from the topological nature of the chiral superconductor. As a result, a quasiparticle state that is exactly at zero energy will be found. The details of obtaining this state within a particular model of the vortex are below.

Once again we consider a circularly symmetric vortex with a step profile for the magnitude of the pair potential. The BdG Hamiltonian in the core region $(r < L)$ is simply the kinetic energy term:

$$\mathcal{H} = \frac{1}{2m} \left( \partial_x^2 + \frac{\hbar^2}{4r^2} - p_x^2 \right) \sigma_z$$  \hspace{1cm} (58)

where $\delta = -i\hbar \partial_r$ is the angular momentum operator, and we have introduced $\delta r = -i\hbar \partial_r + \frac{i \hbar}{r}$, which, in the combination in square brackets above, reduces to the radial part of the Laplacian:

$$\partial_r^2 - \frac{\hbar^2}{4r^2} = -\hbar^2 [\partial_r^2 + \frac{1}{r} \partial_r]$$

The Hamiltonian in the region outside the core $(r > L)$ takes the form:

$$\mathcal{H} = \frac{1}{2m} \left( p^2 - p_F^2 \right) \sigma_z + \frac{\Delta_0}{p_F} e^{i \delta \theta} (p_x - i p_y) e^{-i \delta \theta} \sigma^z + h.c.$$  \hspace{1cm} (59)

where the phase variation of the pair potential has been written in such a way as to maintain gauge invariance. In cylindrical coordinates, this Hamiltonian takes the simple form:

$$\mathcal{H} = \frac{1}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{\hbar^2}{4r^2} \right) \sigma_z + \frac{\Delta_0}{p_F} \delta \sigma_z + \frac{\Delta_0}{p_F} \frac{l}{r} \sigma_y$$  \hspace{1cm} (60)

To obtain the energy zero eigenstate we look at the quasiparticle states with zero angular momentum. Then, the solution in the region $r < L$ is obvious, it is the Bessel function of order zero ($J_0$) times any constant spinor $\chi$.

$$\psi(r < L) \propto J_0(k_F r) \chi$$  \hspace{1cm} (61)

To solve for the zero energy solution outside the core, it is convenient to make the transformation $\psi(r) = \frac{1}{\sqrt{r}} \phi(r)$. Then, the equation for the zero mode spinor field $\phi^{(0)}(r)$ is obtained by the replacement $P_r \rightarrow i\hbar \delta_r$. Thus we have:

$$\left\{ \partial_r^2 + \frac{k_F^2}{4r^2} - 2k_\Delta \partial_r \sigma_y \right\} \phi^{(0)}(r) = 0$$  \hspace{1cm} (62)

where $\hbar k_\Delta = \Delta_0/v_F$. For a change, we will not linearize the equations, but attempt to solve them in full. We need to find solutions to this equation that are well behaved at infinity.

These solutions can be written as:

$$\phi^{(0)}(r) = g(r) e^{\pm \xi r} \left( \frac{1}{i \mp i} \right)$$  \hspace{1cm} (63)

where $g(r)$ satisfies the equation:

$$\left\{ \partial_r^2 + \frac{k_F^2}{4r^2} - \frac{1}{4r^2} \right\} g(r) = 0$$  \hspace{1cm} (64)

Although this is easily solved in terms of Bessel functions, we will only need its asymptotic properties in the $r \rightarrow \infty$ limit. Clearly, the asymptotic solution to (64) is:

$$g(r \rightarrow \infty) \sim Ae^{i \sqrt{k_F^2 - k_\Delta^2} r} + Be^{-i \sqrt{k_F^2 - k_\Delta^2} r}$$  \hspace{1cm} (65)

where A and B are arbitrary constants. Thus, for an asymptotically well behaved solution, we need to pick $\phi^{(0)}(r)$, but then we have a a pair of arbitrary constants from the solution to $g(r)$. Thus, we always have enough freedom to match the wavefunction and its derivative at $r = L$, \(^{4}\) although we do not present the details here.

\(^{4}\)While matching the wavefunctions the step discontinuity in the pair potential needs to be handled carefully, for example, by using a symmetrized form for the pair potential term $\Delta_0(r) \partial_r \rightarrow \frac{1}{2} \{\Delta_0(r), \partial_r\}$
We just note that it is possible to satisfy the matching condition:

$$\partial_r \log g(r)|_{r=L} = k \frac{J_1(kF L)}{J_0(kF L)}$$  \hspace{1cm} (66)$$

Thus, there is always a zero energy eigenstate present in this problem. Notice, that this is the case even if $\Delta > E_F$, when one of the solutions of $g(r)$ is now an increasing function with distance, but convergence is still obtained due to the exponential prefactor $e^{-kF r}$. As is typical in these situations, the particle hole transformation for spin triplet superconductors leaves this zero energy state unchanged. We then have a single zero energy state at the vortex core, and as argued earlier, such a state is stable to perturbations due to particle hole symmetry. A relevant perturbation arises if there are other zero modes in the vicinity, such as if there is a another vortex nearby. In that case the states in the two vortices can tunnel and give rise to a small splitting that is exponentially small in the separation. It is interesting to consider the fate of these zero energy states in the strong pairing regime $E_F < 0$. Then, the asymptotic behaviour (64) is now of the form:

$$g(r \to \infty) \sim A e^{-\sqrt{|kF|^2 + k^2} r} + B e^{+\sqrt{|kF|^2 + k^2} r}$$

of which only the first term gives a normalizable solution. Since this leaves us with only a single degree of freedom in choosing this solution, it is not possible in general to satisfy both the wavefunction and derivative matching condition at $r = L$. Therefore, we do not expect zero energy states for strong pairing. As the chemical potential drops through zero, the bulk supports gapless excitations, which release the zero modes in the vortex core present in the weak pairing pairing.

For a $p_x + i p_y$ superconductor in the $S_z = 0$ pairing state, this zero mode of the BdG equations implies that the vortices carry an $S_z$ quantum number of $\pm \frac{\hbar}{2}$. For the case of pairing of spin polarised fermions, the BdG zero mode implies that there is a Majorana fermion associated with the vortex. As a result, the vortices obey an exotic generalization of statistics termed non-Abelian statistics. In contrast to the usual statistics, where interchange of particles leads to a phase change of the wavefunction (anyons), the exchange of non-abelions is represented by a unitary operator. Consider such a superconductor with 2n vortices that are well separated from each other. Due to the Majorana fermion zero modes in each vortex (which are equivalent to n Dirac levels), the ground state is $2^n$ fold degenerate. Then, it turns out that the exchange of vortices is represented by a non-trivial unitary transformation on the space of these degenerate ground states. That is, the vortices are non-abelions. The possibility of non-abelian statistics was raised first in the context of the Read-Moore state$^8$, and the equivalence with the spin polarised $p_x + i p_y$ superconductor was discussed in$^{8,9}$. An easily accessible discussion of non-abelian statistics is in$^{10}$.

III. SEMICLASSICAL DYNAMICS OF SUPERCONDUCTOR QUASIPARTICLES

As we have seen in the previous section, superconductor quasiparticles exhibit a rich variety of behaviour that is very different from the physics of electrons that we are used to. Given this highly unusual behaviour of quasiparticles, it is useful to gain an intuitive understanding of their dynamics, which is the primary goal of this section. To do this, we first develop a semi-classical approximation to the BdG equations, which is then applied to various specific situations. All of these examples were discussed earlier in Section II, within the full quantum theory. Comparison against the known results will serve as a check on the semi-classical approximation, which turns out to agree very well in these specific situations. Since these cases are simple enough to be discussed analytically, and given the pedagogical nature of this article, they are presented in full detail below.

While there are several semiclassical (or quasiclassical) approximations to be found in the literature on superconductor quasiparticles$^{14,15}$, they do not always clarify the underlying physics. The approach described here, we believe, provides a direct, visual understanding of quasiparticle behaviour in different situations. Since the aim of this section is to provide a better appreciation of quasiparticle dynamics, we do not attempt to turn this semiclassical approximation into a quantitative tool by establishing, for example, a precise domain of validity. Rather, we take the agreement of the semiclassical scheme with the full solutions of the examples below as validation for the essential correctness and usefulness of this approach. It must be noted however, that the set of solutions considered are special in that they can be written as the product of a spatially varying wavefunction times a constant spinor. The high accuracy of the semiclassical results is these cases is presumably related to this fact. More generally, for spatially varying spinor wavefunctions, a bigger departure from the full solution is expected. A more rigorous discussion is planned in$^{42}$.

The basic idea behind this semiclassical approximation is now described. As we all know, on taking the classical limit of the Schrödinger equation for electrons, Newtonian mechanics is obtained. Here we ask the analogous question for superconductor quasiparticles. What is the classical limit of the BdG equation? As we shall see, due to the two component nature of the quasiparticles that arises from electron hole mixing, the ‘classical’ quasiparticle has an additional degree of freedom, a unit vector which will be called the Nambu vector $(\eta)$. This Nambu vector captures the electron/hole character of the quasiparticle, for example, its $z$ component represents how electron or hole like the quasiparticle is $(n_z = 1$ electron, $n_z = -1$ hole), while the angle in the $n_x - n_y$ plane is a measure of the relative phase between the electron and hole parts. The classical equations of motion involve a seemingly complex interplay between the position, the
momentum and the Nambu vector of the quasiparticle. When analysed in specific situations however, it will become clear as to how the expected behaviour of quasiparticles, such as Andreev reflection, appear from these equations of motion. Once the classical mechanics is obtained, we derive an expression for the phase acquired by the quasiparticle along a trajectory. This leads to a Bohr-Sommerfeld like quantization condition, that can be used to obtain quasiparticle bound states. The presence of the Nambu vector leads to an additional Berry phase, equal to half the solid angle it sweeps out on the sphere during its motion (same as that for spin half). This Berry phase plays an important role in many of the examples discussed below.

The layout of this section is as follows. First, a path integral representation of the BdG equations is derived, using spin coherent states to account for the two component quasiparticle wavefunction. Since quasiparticle physics is most naturally discussed in the London gauge, where the phase variation of the order parameter is transformed away, the path integral representation for that gauge is derived, with due attention being paid to the presence of vortices. The classical dynamics is then obtained by evaluating the path integral in the stationary phase approximation. The phase associated with the classical paths and a Bohr-Sommerfeld like quantization condition are found, which completes the derivation of our semiclassical approximation. We then apply this to several problems including Andreev reflection and the zero mode in the π shifted SNS junction. Turning to the case of chiral superconductors, we obtain the chiral mode at the edge of a $p_x + i p_y$ superconductor, which is the analog of the skipping orbits at the edge of an electron gas in a magnetic field. The zero mode in the vortex cores of such a superconductor is also derived within this approximation. Finally the semiclassical approximation is used to understand some properties of superconductor quasiparticles in the presence of disorder. There, the motion of the Nambu vector, and the Berry phase acquired by it, will turn out to be crucial ingredients in obtaining a quantum correction to the zero-energy quasiparticle density of states. This feature gives rise to the novel behaviour of the quasiparticle localization classes, that have been much discussed in the literature recently.

A. Derivation of the Semiclassical Equations

In the quantum mechanics of scalar particles the classical limit is most readily accessed via the path integral representation of the propagator. On applying the stationary phase approximation, the classical path - the path for which the action is an extremum - is selected out. Hence we are also provided with a rationale for why classical mechanics can be formulated in terms of variational principles. Here too, in deriving a semiclassical theory for superconductor quasiparticles a similar route is adopted; the propagator associated with the Bogoliubov-deGennes (BdG) equation is expanded as an integral over different paths. The quasiparticle wavefunction has a two component structure, due to the mixing of particle and hole states in a superconductor, and is formally similar to the wavefunction of a spin 1/2 particle. We exploit this analogy and use the spin coherent states in resolving the identity at each intermediate time step. Since these spin coherent states may be pictured as a vector on the unit sphere, $\hat{n}$ (the Nambu vector), the resulting path-integral involves trajectories of a particle that possess this additional degree of freedom. On evaluating the path integral in the stationary phase approximation, the classical equations of motion are obtained, that involve the position ($\vec{r}$), momentum ($\vec{p}$) as well as the unit vector $\hat{n}$. As will be seen, the classical dynamics strongly couples all these degrees of freedom.

For definiteness consider the simple case of the time independent BdG Hamiltonian of an s-wave superconductor in a magnetic field:

$$
\hat{H}_{BdG} = \frac{1}{2m} (\vec{p} - e \vec{A}(r) \vec{\sigma})^2 - \mu |\sigma_{+} + \Delta_{R}(r) |\sigma_{-} + \Delta_{Im}(r) |\sigma_{\pi} \right)

(67)

where $\Delta_{R}$ and $\Delta_{Im}$ are the real and imaginary parts of the pair potential, and $\vec{A}$ is the vector potential which are all assumed as given. The two component nature of the quasiparticles wavefunction arises from electron hole mixing in the superconductor. Thus, in the quasiparticle wavefunction $\psi(r) = [u(r) \, v(r)]^T$, the electron amplitude is $u(r)$ and the hole amplitude is $v(r)$. In Dirac notation, if we denote $|e\rangle \rightarrow [1, \, 0]^T$ and $|h\rangle \rightarrow [0, \, 1]^T$ we have,

$$
|\psi\rangle = \int d^3r \ u(r)|r\rangle|e\rangle + v(r)|r\rangle|h\rangle

(68)

and the propagator is given by

$$
\hat{K}(t > 0) = e^{-i \hat{H}_{BdG} t}

(69)

K_{\epsilon i}(r', r; t) = \langle i' | (r') e^{-i \hat{H}_{BdG} t} | r\rangle |i\rangle

(70)

where $i, i' \in \{e, h\}$.

A path integral representation of this propagator is derived in the standard fashion. First divide the time interval $t$ into $N$ equal time steps ($\epsilon = t/N = t_{j+1} - t_j$) and use the resolution of the identity to obtain:

$$
K_{\epsilon i}(r', r; t) = \sum_{i_1 = (e, h)} \cdots \sum_{i_{N-1} = (e, h)}

\int d^3r_1 d^3r_{N-1} \prod_{j=0}^{N-1} K_{i_{j+1} i_j}(r_{j+1}, r_j; \epsilon)

(71)

where we have set $(r_0, i_0) = (r, i)$ and $(r_N, i_N) = (r', i')$. While this formulation that involves a 2x2 matrix at each time step may be pursued further with profit, we adopt
a different approach that leads to a more intuitive appreciation of quasiparticle dynamics. The key feature of this approach is that it utilizes the coherent state representation of spin 1/2, that applies equally to our two component quasiparticles. Details of this coherent state representation are described in Appendix A. Here we just note that it is possible to form a linear combination of electron and hole states that is along the unit vector \( \hat{n} \) in Nambu space. The set of such states \( |\hat{n}\rangle \) forms an overcomplete basis and can be used to resolve the identity within this two dimensional subspace:

\[
\int \frac{d\hat{n}}{2\pi} |\hat{n}\rangle \langle \hat{n}| = 1
\]  
(72)

This identity is used to expand the propagator (70) in terms of a path integral over these states. This yields:

\[
K_{\hat{r}i}(r', r; t) = \int \frac{d\hat{n}}{2\pi} \frac{d\hat{n}'}{2\pi} \langle \hat{n}' | \hat{n} \rangle \langle \hat{n} | \hat{n}' \rangle \left[ \int D\hat{n} D\hat{n}' \prod_{j=0}^{N-1} K(r_{j+1}, \hat{n}_{j+1}; r_j, \hat{n}_j; t) \right] 
\]

where the propagator for infinitesimal time evolution in the coherent state representation is given by:

\[
K(r_{j+1}, \hat{n}_{j+1}; r_j, \hat{n}_j; t) = \langle \hat{n}_{j+1} | (r_{j+1}) e^{-iH_{\text{int}} t} | r_j, \hat{n}_j \rangle 
\]

Thus, the term within square brackets in (73) is the propagator in the coherent state basis, which is related to the propagator in the particle-hole basis via the wave function factors \( \langle \hat{n}' | \hat{n} \rangle \langle \hat{n} | \hat{n}' \rangle \). Henceforth we shall just focus on the propagator in the coherent state basis. It’s path integral representation can be obtained in the usual way, the details of which are in Appendix B. The result is:

\[
K(r', \hat{n}'; r, \hat{n}; t) = \int D\hat{n} D\hat{n}' Dr p \ e^{iS/\hbar} 
\]

where

\[
S = S_B - \int dt \mathcal{H}(r, p, \hat{n}) 
\]

\[
S_B = \int dt p \frac{dr}{dt} + \frac{\hbar}{2} \int dt \left[ 1 - \cos \theta(t) \right] \frac{d\phi}{dt} 
\]

\[
\mathcal{H}(p, r, \hat{n}) = \left[ \frac{p^2 + e^2 A^2(r)}{2m} - \mu |n_z| - \frac{e}{m} [p \cdot A(r)] \right] + \Delta_R(r) n_x + \Delta_{I\text{m}}(r) n_y 
\]

the second term on the right hand side of equation (77) is just the Berry phase for spin one half and is equal to half the solid angle swept out by the tip of the Nambu vector \( \hat{n} \). In (77) it is expressed in terms of the angular coordinate of \( \hat{n} \) by introducing the vector potential of a unit monopole located at the center of the sphere: \( \hat{A}(\hat{n}) \). Then, by Stokes theorem the solid angle \( \Omega \) enclosed is just:

\[
\Omega[\hat{n}] = \int (1 - \cos \theta) d\phi = \int \hat{A}(\hat{n}) \cdot d\hat{n}
\]

(79)

which is another convenient representation of the Berry phase term. In contrast to the usual case of a scalar particle where the momentum is integrated out to yield a true ‘path’ integral over just the coordinates and their time derivatives, here we choose not to carry out this integration since the kinetic term is not purely quadratic in the momentum but also involves \( n_z \). However, this will not pose much of a problem since we are primarily interested in the semiclassical limit.

We now proceed to evaluate (75) in the stationary phase approximation which will lead to the classical equations of motion. However, we postpone this step and first derive the propagator in a special gauge (the London gauge), and then make the classical approximation. The advantage of working in the London gauge is that the resulting classical equations admit to a more transparent physical interpretation.

1. The London Gauge

Given a pair potential \( \Delta(r) = |\Delta(r)| e^{i\Phi(r)} \) it is possible to eliminate the phase \( \Phi \) by making a suitable gauge transformation. In this new gauge, called the London gauge, the s-wave pair potential is real everywhere. The unitary operator

\[
U = e^{-\frac{i}{\hbar} \Phi(r)} \sigma_z 
\]

implies this gauge transformation. Thus the Hamiltonian in the London gauge is,

\[
\hat{H}_L = U \hat{H}_{\text{BdG}} U^{-1} 
\]

\[
= \left[ \frac{1}{2m} p + P_s \sigma_z \right]^2 - \mu |\sigma_z| + |\Delta(r)| \sigma_x 
\]

(81)

\[
\psi_L = U \psi_{\text{BdG}} 
\]

(82)

where the vector potential in the London gauge

\[
P_s(r) = \left( \frac{1}{2} \nabla \Phi - eA \right) 
\]

is a gauge invariant physical quantity that represents the momentum carried by each member of a Cooper pair at point \( r \). Note, that in the presence of elementary \( hc/2e \) vortices, the unitary operator \( U \) is not single valued, and so the transformed wavefunction \( \psi_L \) possesses branch cuts emanating from the positions of the vortices, across which it changes sign.

In deriving the path integral representation of the propagator in the London gauge we use a rotated (gauge transformed) basis at each time step. Rather than the transformation (80) it is more convenient to
use the single valued unitary transformation \( U(r) = e^{i\Phi(r)} e^{-i\Phi(r)} \sigma_z \); where for definiteness we stick to the positive sign choice\(^{13}\). Then, the rotated basis states \( \tilde{U}^{-1}(r) |\tilde{n}\rangle |r\rangle \) are used to expand the propagator. A careful analysis yields:

\[
K_L(r', \tilde{n}; r, \tilde{n}; t) = \int \mathcal{D}\tilde{n}' \mathcal{D}p_e e^{i S_L'/\hbar} \\
S_L = S'_L - \int \mathcal{H}_L dt
\]

where

\[
\mathcal{H}_L(r, p, \tilde{n}) = \left[ \frac{p^2 + P_z^2}{2m} - \mu \right] n_z + \frac{P_r(r)}{m} |\Delta(r)| n_z \quad (83)
\]

\[
S'_L = \int p \cdot dr + \frac{\hbar}{2} \int \mathcal{A}(\tilde{n}) \cdot d\tilde{n} + \frac{\hbar}{2} \int \frac{d\Phi}{dt} dt \\
(84)
\]

The last term on the right hand side of equation (86) is a total derivative and so does not affect the classical equations of motion. However in the presence of \( \hbar c/2e \) vortices, the only case where it makes a contribution, it gives rise to the required phase factor of (-1) for paths circling the vortex.

2. Classical Equations of Motion

The path integral (83) can be evaluated in the stationary phase approximation which selects out the path that is an extremum of \( S_L \), that is, the classical path. Thus the following classical equations of motion for the quasiparticle are obtained:

\[
\frac{\delta S_L}{\delta \tilde{n}} = 0 \Rightarrow \\
\frac{d\tilde{n}}{dt} = \frac{\mathcal{A}(\tilde{n})}{\hbar} \\
\frac{d\tilde{r}}{dt} = \frac{\mathcal{A}(\tilde{n})}{\hbar} n_z + \frac{\p_r(r)}{m} \\
\frac{\delta S_L}{\delta \tilde{r}} = 0 \Rightarrow \\
\frac{d\tilde{r}}{dt} = \bar{\sigma} \cdot \mu(r) n_z \\
- \frac{\delta}{\delta \tilde{n}} \left[ \frac{p \cdot p_e(r)}{m} + \frac{P_z^2(r)}{2m} n_z + |\Delta(r)| n_z \right] = 0 \\
S_L = 0 \Rightarrow \\
- \frac{\hbar}{2} \frac{d\tilde{n}}{dt} = (|\Delta(r)|, 0, \frac{p^2 + P_z^2(r)}{2m} - \mu) \times \tilde{n} \\
(88)
\]

Clearly, these equations of motion can be derived from the classical Hamiltonian (85), by assuming appropriate Poisson brackets between \( r \) and \( p \), as well as between the components of \( \tilde{n} \). These equations of motion admit to a simple interpretation. First consider the velocity equation (87). The first term on the right hand side of the equation relates the momentum to the velocity and conforms to our expectations in the two extreme cases when the quasiparticle is a pure electron or a pure hole. For an electron \( (n_z = +1) \) with momentum \( p \) we of course expect the velocity to be \( p/m \) while for a hole \( (n_z = -1) \) with the same momentum the velocity would be reversed. The second term on the right hand side is just the superfluid velocity which plays the role of the reference frame with respect to which the quasiparticles move. Next, consider the force equation (88). If for a moment we assume that the chemical potential is a function of space \( \mu(r) \), then the first term on the right hand side of (88) is just the gradient of the chemical potential time the ‘charge’ \( n_z \) of the quasiparticle. The other terms arise from inhomogeneities in the pair potential and superflow. Finally the motion of the Nambu vector \( \tilde{n} \) as described by equation (89) is the same as that of a spin precessing in a Zeeman field of \( \mathcal{J} = -\frac{\hbar}{2} |\Delta(r)| \), \( 0 \), \( \frac{\hbar}{2m} [p^2 + P_z^2(r)] - \mu \). This Nambu-Zeeman field depends on both the position and momentum of the quasiparticle and the motion of the quasiparticle in turn is strongly affected by the Nambu vector. This then is the classical limit of the BdG equations, just as Newton’s mechanics are recovered from the classical limit of Schrödinger’s equation.

Shortly, these equations of motion are solved for some simple physical situations. Then it will become apparent how this seemingly complex interdependence of the degrees of freedom lead to the characteristic behaviour of superconductor quasiparticles such as Andreev reflection. Before that we turn to deriving an analogue of the Bohr-Sommerfeld quantization rule which when used in conjunction with the classical dynamics allows us to quantize the quasiparticle states in some simple situations.

3. The Quantization Condition for Quasiparticles

In order to develop a semiclassical quantization scheme for quasiparticles, we need to supplement the classical equations of motion with a quantization condition like that of Bohr and Sommerfeld. It is not hard to guess this condition; a detailed derivation from the path integral (83) is presented in Appendix C. The result is:

\[
S_{B,cl}(r_{cl}, p_{cl}, \tilde{n}_{cl}) = \oint p_{cl} \cdot dr_{cl} + \frac{\hbar}{2} \oint \mathcal{A}(\tilde{n}_{cl}) \cdot d\tilde{n}_{cl} \\
+ \frac{\hbar}{2} \oint d\Phi(r_{cl}) = 2\pi n \hbar \\
(90)
\]

where the phase acquired along the classical path \( S_{B,cl} \) has, in addition to the usual Bohr-Sommerfeld phase of \( \oint p \cdot dr \), a Berry phase term arising from the motion of the Nambu vector. Also, there is the term that accounts for the phase change of (-1) on circling an elementary vortex. The quantization condition requires the sum of all these phases on one traversal of a periodic classical orbit to be an integer multiple of \( 2\pi \).
4. Non s-wave Pairing

We now briefly comment on how the semiclassical equations are modified when one considers pairing that is more complex than the s-wave pairing discussed so far. As shown previously, the quasiparticles of all the singlet pairing states and some triplet pairing states as well are governed by 2x2 BdG equation, but now the pair amplitude seen by the electrons is a function of their momentum i.e. $\Delta \rightarrow \Delta(p)$ for the homogenous problem in equation (9). For the inhomogenous problem, when $\Delta$ is a function of both position and momentum, issues of operator ordering arise. Since the different orderings differ by terms that contain an additional power of $\hbar$, we can ignore this feature in the classical approximation. Thus, the classical Hamiltonian is given by:

$$H_{cl} = \left[ \frac{p^2}{2m} + F_p(r) \right] - \mu]n_z + \frac{1}{m} p \cdot P_e(r) + \Delta_R(p, r)n_x + \Delta_I_m(p, r)n_y$$

(91)

where $\Delta$ is an even (odd) function of $p$ for a singlet (triplet) superconductor. In the Hamiltonian above is written in the London gauge, where the spatial phase variation of the order parameter has been eliminated by a suitable gauge transformation. In contrast to the s-wave case, it is not always possible to obtain a potential that is real, even after such a transformation. The equations of motion that result are:

$$\frac{d\vec{r}}{dt} = \frac{\vec{p}}{m}$$

(92)

$$\frac{d\vec{p}}{dt} = -\vec{\partial}_p H_{cl}$$

(93)

$$-\frac{\hbar}{2} \frac{d\vec{n}}{dt} = (\Delta_R, \Delta_I_m, \frac{1}{2m}[p^2 + F_p^2(r)] - \mu) \times \vec{n}$$

(94)

the new feature is that there are additional velocity terms arising from the variation of the gap with momentum. Thus, unlike in the case of electrons, the relation between velocity and momentum for quasiparticles is far from straightforward, which leads to some of the novel behaviour that arise in these systems.

B. Applications of the Semiclassical Theory

We now apply the semiclassical theory that we have derived to various simple situations. Although the full quantum mechanical solutions to these are known from solving the Bogoliubov-deGennes equation, the semiclassical treatment provides some insights into the dynamics of quasiparticles, as well as a simple way of accessing the quantum mechanical results. Besides, the agreement serves as a check on the semiclassical treatment. We begin by considering the simplest possible case - that of quasiparticles in a homogenous superconductor, before moving on to progressively more involved cases, paralleling the discussion in Section II. Again, quasiparticle states at zero energy will be given special attention, since these cases are straightforward enough to show in full analytical detail, as described below.

1. Free Superconductor Quasiparticle

Consider a quasiparticle in a homogenous s-wave superconductor, governed by the Hamiltonian:

$$H_{free} = \left( \frac{p^2}{2m} - \mu \right) \sigma_z + \Delta \sigma_x$$

(95)

Then, the classical motion is governed by the Hamiltonian:

$$H_{cl} = \left( \frac{p^2}{2m} - \mu \right)n_z + \Delta n_x$$

(96)

which yields the classical equations of motion:

$$\dot{p} = 0$$

$$\dot{r} = \frac{p}{m}n_z$$

$$-\frac{\hbar}{2} \dot{\vec{n}} = (\Delta_R, \Delta_I_m, \frac{1}{2m}[p^2 + F_p^2(r)] - \mu) \times \vec{n}$$

Where the dot denotes the time derivative. For simplicity imagine the motion to be taking place in one spatial dimension with periodic boundary conditions (circle). The momentum $p$ is a constant of the motion, while the Nambu vector $\vec{n}$ precesses about the 'Zeeman' field $\vec{f}_p = -\frac{\hbar}{2}(\Delta, 0, \frac{p^2}{2m} - \mu)$. For a given momentum, the component of the Nambu vector along $\vec{f}_p$ is also an independent conserved quantity $n_f = \vec{n} \cdot \vec{f}_p/|\vec{f}_p|$. Since this system is integrable, it is appropriate to use the EBK (Einstein-Brillouin-Keller) quantization scheme. In simple terms, this amounts to the following. The motion of an integrable system can be decomposed into a set of periodic motions. The EBK quantization imposes the Bohr-Sommerfeld conditions independently, on each of these separate motions.

Let us begin by quantizing the motion of $n_f$ and its conjugate variable, $\phi_f$, the azimuthal angle about this

---

5This is most easily seen by transforming to action-angle variables. An integrable system with $n$ degrees of freedom has $n$ constants of motion (actions) $I_1, \ldots, I_n$ that have vanishing Poisson Brackets with each other. The Hamiltonian is solely a function of these variables $H = H(I_1, \ldots, I_n)$. Thus, the equations of motion of the conjugate variables (angles) $\Phi_1, \ldots, \Phi_n$ follow $\Phi_i(t) = \omega_i t + \Phi_i(0)$ where the constant frequencies are given by $\omega_i = \frac{\partial I_i}{\partial I_i}$. Hence the motion can be decomposed into a set of independent periodic motions.
axis. Then, the Bohr-Sommerfeld phase (90) acquired on a closed path on the Nambu sphere is half the solid angle $\Omega$ swept out, which in these coordinates are:

$$ \frac{1}{2} \Omega[n] = \frac{1}{2} \oint (1 - n_f) d\phi_f = \pi(1 - n_f) $$

(97)

The Bohr-Sommerfeld condition then requires $n_f = \pm 1$, that is the Nambu vector is either parallel or anti-parallel to the field $f_e$. The two solutions yield the familiar expression for the quasiparticle energy:

$$ E_p = \pm \sqrt{\left( \frac{p^2}{2m} - \mu \right)^2 + \Delta^2} $$

(98)

to quantize the other conserved quantity, the momentum, we note that the quasiparticle has a velocity ($n_z = \frac{p}{m} \frac{e^2}{\hbar} \frac{\Delta}{E_F - \mu}$) that causes it to drift around the circle. Then, the total phase acquired in this motion should also be quantized:

$$ \oint p \, dx = pL = n2\pi\hbar $$

(99)

We then have the usual quantization condition on the momentum $p = n \frac{2\pi\alpha}{L}$, where $L$ is the length of the circle. Thus we have recovered the correct eigenstates of a quasiparticle with periodic boundary conditions. In this case the semiclassical approach is more involved than a direct solution of the BdG equation. If this were generally true, the semiclassical theory would be of limited utility. As shown in the less trivial examples below, this is not always the case.

2. Andreev Reflection

Here we consider the situation when a normal metal is in contact with a superconductor in one dimension. Since the motion is unbounded, the quantization conditions are not imposed. Rather we simply study the classical motion to gain some insight into the process of Andreev reflection. We will work with a step pair potential, i.e. an abrupt transition from the metal ($x < 0$) to the superconductor ($x > 0$).

$$ \Delta(x) = \Delta_0 \text{ for } x > 0 $$

$$ = 0 \text{ for } x < 0 $$

A more natural pair potential that varies smoothly, can be argued to give similar results.

**The discontinuity in the pair potential is handled by considering it as the limit of a smooth profile. That the momentum remains constant with these initial conditions can then be easily verified.**

FIG. 4. An electron in the metal with momentum $p_F$ moving towards the superconductor (initial state). The Nambu vector $\hat{n}$ and velocity are shown. On entering the superconductor, the Nambu vector rotates in the $n_x, n_y, n_z$ plane due to the pair potential $\Delta_0$. The velocity (which is proportional to $n_z$) decreases, till the quasiparticle stops when the Nambu vector is along $n_y$. Since the rotation continues, the velocity reverses and when the quasiparticle leaves the superconductor, the Nambu vector is along $(0, 0, -1)$ (a hole). It is easily checked that the momentum is constant during this motion. Hence, an electron with momentum $p_F$ is reflected as a hole with the same momentum but reversed velocity - which is Andreev reflection. The incoming and outgoing parts of the trajectory are shown, for clarity, as displaced in the vertical direction.

Consider an electron in the metal that moves towards the superconductor. We want to follow its subsequent motion using the classical mechanics developed. In order to present all details analytically, we consider the special situation when the electron is exactly at the Fermi energy, for which the analysis simplifies considerably. We thus begin with the initial conditions $\hat{n}(0) = (0, 0, 1)$ [electron], $\hat{p}(0) = p_F \hat{x}$ and $x(0) < 0$. Integrating the equations of motion:

$$ \dot{x} = \frac{p(t)}{m} n_z(t) $$

$$ \dot{p} = -\frac{d\Delta}{dx} n_x(t) $$

$$ -\frac{\hbar}{2} \hat{n} = (\Delta(x), 0, \frac{p^2}{2m} - \mu) \times \hat{n} $$

it turns out that for these initial conditions that $n_z(t) = 0$ throughout the motion, a fact we will verify at the end of the analysis. As a consequence, the momentum remains constant $p(t) = p_F \hat{x}$ ** and we are left with solving the pair of equations:

$$ \dot{x} = v_F n_z(t) $$

$$ -\frac{\hbar}{2} \Delta \hat{n} = (\Delta(x), 0, 0) \times \hat{n} $$

where $v_F = p_F/m$ is the Fermi velocity. The quasiparticle moves with velocity proportional to the component of the Nambu vector $n_z$, and the Nambu vector in turn is rotated by the pair potential about the $n_z$ direction. The motion then is as shown in figure (4). On entering the superconductor, the Nambu vector is rotated in the $n_x, n_y$ plane till it is completely along the $n_y$ direction. At this point the quasiparticle velocity is zero. However, the Nambu vector continues to rotate and now takes on
negative $n_z$ values. The quasiparticle velocity is therefore reversed and the path in real space is retraced till it exits the superconductor. At that point the Nambu vector has completed a $\pi$ rotation and now is $(0, 0, -1)$ i.e. the electron has returned from its excursion into the superconductor as a hole. Although the momentum stayed constant throughout, the hole moves with a reversed velocity as seen from the velocity equation above. This is just Andreev reflection within the classical formulation. The classical trajectory is given by:

$$x(t') = \frac{\xi}{2} \sin \omega t'$$

$$p(t') = p_F$$

$$\dot{n}(t') = (0, \sin \omega t', \cos \omega t')$$

where $\omega = 2 \Delta_0 / \hbar$, $\xi = v_F \hbar / \Delta_0$ and $t'$ is measured from the time the quasiparticle enters the superconductor and $0 < t' < \pi / \omega$. Since the motion of the Nambu vector occurs in the $n_z n_x$ plane, our assumption of $n_x = 0$ throughout the motion is shown to hold. It is amusing to compare this classical solution with the solution to the BdG equation in Section II B2. For example, the classical particle penetrates a distance $\frac{\hbar \xi}{m} = \xi / 2$ which can be compared with the decay length of the evanescent wave that is present in the BdG solution. This is rather different from the case of ordinary particles obeying the Schrodinger equation, where evanescent waves occur in regions that are classically forbidden, and the classical mechanics contains no hint of them.

Finally, we note that identical results are obtained for a smooth profile of $\Delta(x)$ as long as we begin with the same initial conditions in a region with $\Delta = 0$. The case of quasiparticles with an energy different from zero can be similarly solved although some of the simplifying features above are lost.

3. S-N-S Junction

Consider now a normal region sandwiched between two superconductors with a relative phase difference $\Phi$. Assuming a sharp transition between these regions, we have the pair potential:

$$\Delta(x) = \begin{cases} 
\Delta_0 e^{i\phi} & x < -L \\
0 & -L < x < L \\
\Delta_0 & L < x 
\end{cases}$$

Again, we analyze the motion of the zero energy state. Begin with an electron in the normal region with momentum $p_F \hat{x}$. As it turns out, the momentum remain a constant for this motion as well. The equations of motion then reduce to

$$\dot{x} = v_F n_z(t)$$

$$-\frac{\hbar}{2} \dot{n} = (\Delta_R(x), \Delta_{lm}, 0) \times n$$

where $\Delta(x) = \Delta_R(x) + i \Delta_{lm}(x)$. The classical motion with these initial conditions is as shown in figure 5. The electron enters the right superconductor from which it emerges as a hole, just as in the previous section on Andreev reflection. It then enters the left superconductor, where it is rotated back into an electron and re-emerges in the metallic region. However, the plane of rotation of the Nambu vector in the two superconductors is different due to the relative phase difference $\Phi$ between them, as shown in figure 5b. While the Nambu vector rotates in the $n_z n_x$ plane in the right superconductor, in the left superconductor it rotates in the $n_z (\sin \Phi n_x - \cos \Phi n_y)$ plane, so as to stays orthogonal to the pseudo-Zeeman field $(\Delta_R, \Delta_{lm}(x), 0)$. Since the classical equation closes on itself after a time period $T$, that is, we have $[x(T), p(T), \dot{n}(T)] = [x(0), p(0), \dot{n}(0)]$, we can apply the quantization condition for quasiparticles to ask if there is a bound state at zero energy. The appropriate condition is

$$\oint p \, dx + \frac{\hbar}{2} \Omega[n] = 2\pi m \hbar$$

(100)

where $\Omega$ is the solid angle swept out by the Nambu vector and $m$ is an integer. Since the momentum is a constant throughout, and the motion closes on itself ($\oint dx = 0$), we find remarkably that the first term in the phase above vanishes. Thus only the Berry phase term from the Nambu vector contributes. From the figure 5 it is easily seen that the solid angle swept out is

$$\Omega = 2(\Phi - \pi)$$

(101)

The quantization condition then requires that the phase difference between the two superconductors should satisfy $\Phi = 2\pi m + \pi$. That is, we need a $\pi$ shifted SNS junction to realize a zero energy quasiparticle state. Then, the Nambu vector simply retraces its path and no solid angle is enclosed.
4. Edge States in a $p_x + ip_y$ Superconductor.

As we have seen in Section II C 4, there is a chiral mode at the edge of a $p_x + ip_y$ superconductor. Here we use the classical mechanics for quasiparticles to gain insight into the origin of these edge modes. For the case of electrons exhibiting the integer quantum Hall effect, the edge modes can be pictorially understood as the skipping orbits at the walls enclosing the electron gas, arising from the circular motion of the electrons. We shall discover that an analogous picture emerges for the case of quasiparticles in a chiral superconductor.

For definiteness, consider the situation in figure 6. A $p_x + ip_y$ superconductor ($\Delta = \frac{\Delta_0}{p_F}(p_x + ip_y)$) is on the right, and is separated from the wall at $x = 0$ by a region of normal metal ($\Delta_0 = 0$). As before, we use a semiclassical analysis to look at the zero energy quasiparticle states. We find such a classical trajectory that also satisfies the quantization condition, and is endowed with a chirality. Thus, although we will only access a single state of the chiral branch that is present at the edge, the chiral nature is already reflected in the dynamics of this state.

First note that translation invariance along the $y$ direction implies that $p_y$ is a constant of motion. The effect of the rigid wall is to reverse the momentum along the $x$ direction. We consider the case with $p_y = 0$. Then the classical equations of motion derived from the BdG Hamiltonian (53) for $x > 0$ are, setting both the gap strengths to be equal to $\Delta_0$ in this region:

$$\dot{x} = \frac{p}{m} n_z + \frac{\Delta_0(x)}{p_F} n_x$$

$$\dot{y} = \frac{\Delta_0(x)}{p_F} n_y$$

$$\dot{p}_x = -\frac{d\Delta}{dx} \left( \frac{p_x}{p_F} \right) n_x$$

$$-\frac{\hbar}{2} \dot{n} = (\Delta_0(x), 0, \frac{p^2}{2m} - \mu) \times \hat{n}$$

FIG. 5. (a) Quasiparticle motion in an SNS junction with relative order parameter phase of $\Phi$. Starting with an electron at the Fermi energy, two successive Andreev reflection processes return the electron to its initial state. The Nambu vector and velocity at different points of the trajectory are shown. In (b) the motion of the Nambu vector is shown. Only for the case of $\Phi = \pi$ is the net Berry phase accumulated by the Nambu vector zero, which implies the existence of a (pair of) zero energy states in $\pi$ shifted SNS junctions.

FIG. 6. The zero energy state at the edge of a $p_x + ip_y$ superconductor. For convenience, a normal region of size $L$ is assumed between the wall and the superconductor. The quasiparticle is alternately Andreev reflected and reflected from the wall. The trajectories after reflection from the wall are shown as slightly displaced along the $y$ direction for clarity. However, there is a drift of the quasiparticles along $y$ when in the superconductor, due to the chiral nature of the pairing. The motion in the $x$ direction is exactly as in the $\pi$ shifted SNS junction; the Nambu vector simply retraces its path and the quantization condition is satisfied.
The momentum dependence of the gap gives rise to additional terms in the velocity equation. If we consider the special set of initial conditions of an electron \([\hat{n} = (0,0,1)]\) in the normal region \([0 < x < L]\) with zero energy and moving to the right \(p_x = p_{x+}\), we have a situation very analogous to that in the previous two sections. From our experience there, we can conclude that the \(n_x\) component of the Nambu vector stays zero through the motion and the momentum is also constant (apart from reflection at the walls). This leads to the simplified set of equations:

\[
\begin{align*}
\dot{x} &= v_F n_z \\
\dot{y} &= v_A n_y \\
\frac{\hbar}{2} \dot{\hat{n}} &= (\Delta_0(x), 0, 0) \times \hat{n}
\end{align*}
\]

where \(v_A = \Delta_0/p_F\) is a drift velocity that arises from the momentum dependence of the gap. Aside from this drift, we recognise the above equations as being identical to those considered for the case of Andreev reflection. The electron emerges from the superconductor as a hole propagating towards the wall. In the process it is displaced along the \(y\) axis since \(n_y\) has a non-zero average during this motion. The hole now encounters the wall and has its momentum reversed \(p_x = -p_{x-}\), and hence moves towards the superconductor. The effective equations for this phase of the motion are the same as above, but the quasiparticle sees the opposite sign of the pair potential which is an odd function of momentum:

\[
\begin{align*}
\dot{x} &= -v_F n_z \\
\dot{y} &= v_A n_y \\
\frac{\hbar}{2} \dot{\hat{n}} &= (-\Delta_0(x), 0, 0) \times \hat{n}
\end{align*}
\]

the hole is then Andreev reflected to an electron - which is just like the \(\pi\) shifted SNS junction, and the Nambu vector retraces its path. As a result, the drift along the \(y\) direction is the same as in the first phase of the motion. This drift is an indication of the chirality of the edge, and reverses direction if we consider a \(p_x - ip_y\) superconductor. After another reflection from the wall, we have \([x, p_x, \hat{n}]\) returning to their initial values. Let us now quantize this motion to ensure that the classical trajectory corresponds to a quantum state. In the spirit of EBK quantization, we ignore the unbounded motion along the edge, since that corresponds to the independent conserved quantity \(p_y\). Therefore, we only ask that the periodic motion in \([x, p_x, \hat{n}]\) satisfy the Bohr-Sommerfeld condition. Since we have two reflections at the wall (whose phases then add to \(2\pi\)), and again the fact that \(\oint dp = 0\), the quantization condition requires that the Berry phase picked up by the Nambu vector be a multiple of \(2\pi\). Clearly this is satisfied in this motion as the Nambu vector retraces its path and so acquires no Berry phase.

A similar analysis for a \(d_{x^2-y^2} + id_{xy}\) superconductor reveals two zero energy states that drift in the same direction, corresponding to the two chiral modes that occur at the edge of such a superconductor.

5. Vortex Mode of a \(p_x + ip_y\) Superconductor.

As a final example, the zero energy state in the core of a \(p_x + ip_y\) vortex, that was obtained by solving the BdG equations in Section II C4, is obtained here from the semiclassical method. Since the details turn out to be so similar to the earlier cases that we have considered, we only sketch how the result is obtained. For a rotationally symmetric vortex the classical Hamiltonian in two dimensional radial coordinates \((r, \theta)\) takes the form:

\[
H = \left[ \frac{1}{2m} p_r^2 + \frac{1}{2mr^2} L^2 - \mu \right] n_z + \frac{|\Delta(r)|}{p_F} [p_x n_x + L n_y]
\]

and the modulus of the pair potential is assumed to turn on outside a core size \(L\); that is \(|\Delta(r)| = \Delta_0\) for \(r > L\) and is zero inside this region. It is convenient to go over to the radial and angular momenta \(p_r = \cos \theta p_x + \sin \theta p_y\) and \(L = r(\cos \theta p_y - \sin \theta p_x)\). Then the Hamiltonian takes the simple form:

\[
H = \left[ \frac{1}{2m} p_r^2 + \frac{1}{2mr^2} L^2 - \mu \right] n_z + \frac{|\Delta(r)|}{p_F} [p_x n_x + L n_y]
\]

The zero energy state turns out to have \(L = 0\). The equations of motion then read:

\[
\begin{align*}
\dot{r} &= \frac{p_r}{m} n_z + \frac{|\Delta(r)|}{p_F} n_x \\
\dot{\theta} &= \frac{\Delta(r)}{p_F r} n_y \\
\dot{p_r} &= -\frac{d|\Delta(r)|}{dr} (p_r/p_F) n_x \\
\frac{\hbar}{2} \dot{\hat{n}} &= \left( \frac{p_r}{p_F} |\Delta(r)|, 0, 1, \frac{1}{2m} p_r^2 - \mu \right) \times \hat{n}
\end{align*}
\]

since these equations are like the ones in the previous section, we do not analyze the motion in detail here. For example, \(p_r\) here plays the role of \(p_x\) in that case, and changes sign when it crosses \(r = 0\), so the center can be visualized as a rigid wall that reflects the quasiparticle and changes the sign of \(p_x\). Using the EBK quantization, the motion in the radial direction is shown to satisfy the Bohr-Sommerfeld condition, and the angular motion quantizes the angular momentum, of which \(L = 0\) is an allowed quantum number.
FIG. 7. Classical trajectory that gives rise to the zero energy state in the vortex of a $p_x + ip_y$ superconductor. Here the vortex circulation is directed opposite to the superconductor chirality. The pair potential is assumed to vanish for $r < 1$ (where distance is measured in units of the coherence length) and $\Delta_0/E_F = 0.4$. The angular drift of this zero angular momentum quasiparticle state arises from the chirality of the superconductor. The motion of the Nambu vector and radial coordinate is similar to that in the $\pi$ shifted SNS junction.

This motion is pictured in figure 7. Note that the angular motion proceeds in a direction that is controlled by the chirality of the superconductor. The motion for an oppositely directed vortex that gives rise to a zero energy state is similar, although in that case the quasiparticle avoids going through the center of the vortex.

C. Weak Localization of Quasiparticles: A Semiclassical Picture.

The study of non-interacting electrons in a disordered medium has attracted much attention in the last several decades. One tractable limit of this problem is when the spacing of impurities is much larger than the Fermi wavelength, and is termed the weak localization regime. There, in the limit of low temperatures, phase coherence between certain pairs of paths leads to quantum corrections to the diffusive behaviour of electrons. The presence of such phase coherent paths arise from the symmetries of the problem and we have three universality classes depending on the presence of time reversal (T) and SU(2) spin (S) symmetries. When T is broken, only a single universality class is obtained, irrespective of whether S is preserved.

Recently, a similar analysis for noninteracting superconductor quasiparticles has been carried out. Since quasiparticles do not carry a fixed electrical charge, the transport quantities of interest are the thermal or spin conductivities. The Bogoliubov-deGennes Hamiltonian that governs the noninteracting quasiparticles has an additional particle-hole symmetry that leads to four new localization universality classes depending on the presence of T and S symmetries. While in the case of ordinary electrons, the quantum corrections only affect the diffusivity, here the particle-hole symmetry gives rise to quantum corrections to the density of states at the quasiparticle chemical potential, and hence affect the conductivity.

Interestingly, a quantitative theory of weak localization for electrons can be developed based on the semiclassical method. There, the quantum corrections to the diffusivity arise from the presence of classical paths that return close to their starting points with reversed momentum. In the presence of time reversal symmetry, the same trajectory can be traversed in the opposite direction and will then acquire the same phase. Due to this phase coherence the probability for such a momentum reversing process is increased over the classical (incoherent) value and leads to a superposition of the diffusivity. In dimensions two and below (when a random walk always returns to its starting point) this coherent backscattering effect is strong enough to always localize the electronic states.

One may ask if a similar approach may be taken for superconductor quasiparticles. In fact, quantitative results for the four superconductor classes that agree with the field theoretical calculations can be obtained using the semiclassical approximation developed in this section. Here we briefly highlight the physics uncovered by this approximation, details will be presented elsewhere.

For the case of quasiparticles with T and S symmetries, there is an exact analog of the coherent backscattering process that leads to a suppression of the diffusivity. In addition, it can be seen that there is a quantum correction to the density of states (DOS) at the quasiparticle chemical potential (zero energy), which may be obtained by the following semiclassical argument. Assume that we start with a finite DOS at zero energy and then consider the effect of quantum corrections. Then, the quasiparticle trajectories of interest are as follows. If a quasiparticle trajectory beginning at $(r, \theta, \phi)$ at $t = 0$ were to return at time $t = T/2$ to $(r, -\theta, -\phi)$, then subsequently the classical motion will retrace the original path and return

\[11\] When S is preserved. If spin rotation symmetry is broken, the motion of spin leads to an additional Berry phase that reverses the sign of the effect.

23
to the initial state at time $t = T$. Such classical paths that return to their initial state figure in the semiclassical approximation to the density of states. In order to have the quasiparticle in the specified states at $t = 0, T/2$, it is easily verified that the classical motion must be at zero energy. Hence the quantum correction to the DOS that we will obtain from this process is restricted to the vicinity of $E = 0$ as we would expect from the particle hole symmetry of the BdG. An analysis of the phase acquired by the quasiparticle on this path reveals that while the Bohr phase ($\oint pdr$) is zero, the Nambu vector sweeps out exactly half the sphere leading to an overall phase of $(-1)$ for the spin rotationally invariant case. This leads to a supression of the density of states. Note, that in this argument for the DOS effect, time reversal symmetry is not invoked and is hence present regardless of $T$. All this is just a restatement of the arguments presented in$^{50, 4, 51}$. There, properties of Green functions arising from the particle-hole symmetry were used to construct a ‘semiclassical’ argument to understand the results obtained in those works. In our semiclassical approximation however, it is also possible to go beyond these basic observations. For one, it is easily seen that the count of the number of semiclassical paths that contribute to the DOS effect is the same as that for the coherent backscattering effect. Hence there is a precise sense in which the localization effect in the T+4 situation, which receives both DOS and coherent backscattering corrections, is twice as strong as when T is broken (but S preserved).

The case when spin rotation symmetry of the quasiparticles is broken can also be analyzed in a similar fashion. The trajectories that affect the DOS now require that the spin vector (also thought of in a coherent state representation) also sweeps out half the sphere, and hence the combined Berry phases of both the spin and Nambu vectors combine to yield a total phase factor of $+1$. Thus, there is an enhancement of the DOS about zero energy in this case. Also, the coherent backscattering process (with T) now leads to an enhancement of the diffusivity due to the absence of S (the familiar weak-antilocalization). Taken together this means that there is a quantum enhancement of the conductivity for the case of quasiparticles with broken S, and the effect is twice as large if T is preserved, by our previous path counting arguments. This then is the semiclassical picture of localization in the four superconducting classes.

Finally we note that the case of a superconductor of spinless fermions with time reversal symmetry is particularly interesting. There, it is easily argued that the $E = 0$ DOS is enhanced, while the coherent backscattering suppresses the diffusivity. By the path counting argument, we expect these two effects to be of equal magnitude and hence to cancel each other when combined to obtain the conductivity.

### IV. DISCUSSION

In this article we have reviewed some of the physics associated with superconductor quasiparticles in the presence of vortices and other topological defects. A semiclassical approximation was introduced with the intention of providing some intuition into the unusual behaviour of quasiparticles. This approximation was applied to a number of simple situations and to understanding the weak localization of quasiparticles in disordered superconductors. The rest of this section is devoted to a discussion of open problems in the area of quasiparticle physics. Several of these problems are related to recent experimental results in the field, while others arise as a natural continuation of the earlier material.

In Section II we discussed the bound states that appear in the vortex core of an s-wave superconductor. For the case of d-wave superconductors, that have nodal points with low energy excitations, no bound state is expected to occur at the core. Contrary to these expectations, Scanning Tunneling Microscopy (STM) on vortices in the high-temperature superconductor YBCO$^{23}$, and more recently BISCO$^{51}$, indicate the existence of a quasiparticle state bound to the vortex core. This state is at a fairly high energy, roughly 5 to 7meV, and hence cannot be explained by assuming a small “is” or “id” component induced by the magnetic field near the core. Another puzzling feature of the vortex bound state is that it shows an exponential fall off with distance from the vortex core that is nearly isotropic, despite the underlying d-wave structure. In addition, the STM spectrum in the core region differs markedly from that calculated from the Bogoliubov-de Gennes equation$^{52}$ for a d-wave superconductor. While the BdG solution predicts a peak in the low energy density of states at the vortex center, exactly the reverse is observed. The vortex core looks very much like the ‘normal’ state (the state above the superconducting transition), where there is a depletion in the low energy density of states due to the pseudogap. All these experimental facts taken together seem to indicate that while the BdG equations describe well the quasiparticles in the cuprate superconductors at longer length scales, this description may break down in the vicinity of the vortex core where the pairing is suppressed and one is forced to confront the strongly correlated nature of the underlying physics. Recently, there have been some theoretical attempts to incorporate these strong correlation effects$^{53}$ using, for example, the slave boson mean field theory and this promises to be an area of theoretical activity in the future.

In discussing the zero energy state in the $\pi$ shifted SNS junction in Section II B3, we noted that a similar effect gives rise to (nearly) zero energy bound states at certain surfaces of a d-wave superconductor. This effect has been observed in tunneling experiments$^{54}$ on optimally doped YBCO. Further, as the temperature is lowered below 8 Kelvin, the zero energy peak is observed to split$^{55}$. The
splitting can be enhanced by the application of a perpendicular magnetic field. In\textsuperscript{56} it was suggested that this was due to spontaneous time reversal symmetry breaking, and the appearance of a subdominant is or id pairing at the surface. However, in\textsuperscript{57} the same effect is ascribed to the dispersion of the one dimensional quasiparticle bands formed near the surface. In view of the theories that invoke a quantum critical point between ‘d’ and ‘d+id’ pairing,\textsuperscript{58} in the cuprate systems to explain the anomalous scaling behaviour of various quantities seen near optimal doping, it would be interesting to confirm or refute the existence of time reversal symmetry breaking at the surface of these systems.

In Section III we discussed a semi-classical approximation to quasiparticle dynamics, mainly with a view to develop an intuition for their novel behaviour. An item for future work would be to turn this into a quantitative tool to study the physics of quasiparticles, for example in mesoscale structures which are currently being actively studied. For the case of electrons, the semiclassical approximation is widely applied in studying mesoscopic phenomena\textsuperscript{59}. Also, issues such as the presence of chaos in quasiparticle classical dynamics and its influence on the spectrum of the BdG equations would be an interesting area of study. Note, that due to the additional degree of freedom provided by the Nambu vector, it is possible that quasiparticle dynamics even in one spatial dimension is chaotic. There are several problems involving quasiparticle transport that could be analyzed by the semiclassical description described here. As a beginning, the Boltzmann equations appropriate for superconductor quasiparticles can be derived in a physically transparent fashion\textsuperscript{42} which can then be applied to the situations of interest. Also, there is a wealth of experimental data on quasiparticle transport in a d-wave superconductor in the vortex state. Explaining this quantitatively remains a major challenge to theory. Another possible application is to the striking effect of de Haas-van Alphen oscillations being seen well below $H_d$ (in the superconducting state) of some conventional superconductors like NbSe$_2$ and some $A-15$ compounds\textsuperscript{80}.

We have briefly discussed the semiclassical understanding of weak localization of superconductor quasiparticles that this approximation provides. In the case of electrons, the semiclassical route is the most direct way to an understanding of the essential physics of localization\textsuperscript{47,48}. For example it provides a ready explanation for the experiments of Sharvin and Sharvin\textsuperscript{61,62}. These experiments, that get to the heart of weak-localization, observe a periodicity of half a flux quantum ($\hbar c/2e$) in the conductance oscillations of a disordered ring, on changing the enclosed magnetic flux. It is hoped that similarly, experiments that highlight the unique features of quasiparticle localization will be realized in the future, and the semiclassical description of quasiparticles could be a valuable aid in the search for such experiments.

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APPENDIX A: COHERENT STATES FOR SPIN ONE HALF

The coherent states $|\hat{n}\rangle$ for spin one half are labelled by the unit vector $\hat{n}$ ($|\hat{n}| = 1$) and are defined by the eigenvalue equation:

$$\hat{n} \cdot \sigma |\hat{n}\rangle = |\hat{n}\rangle$$

where the spin operators $\sigma$ are given by $\sigma = \frac{\hat{z}}{2} \sigma_z$. Thus, for the case of quasiparticles the electron and hole states are defined by

$$|\hat{n} = \hat{z}\rangle = |e\rangle$$

$$|\hat{n} = -\hat{z}\rangle = |h\rangle$$

A coherent state along a general direction $\hat{n}$ parametrized by the angles $\theta, \phi$ with $\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is given by:

$$|\hat{n}\rangle = e^{\frac{i\chi}{2}} e^{\frac{i\varphi}{2}} |\hat{n}\rangle$$

where $\chi$ is an arbitrary phase factor. The coherent states form an overcomplete set of states; their overlap is given by:

$$\langle \hat{n}|'\rangle = e^{\frac{i\chi - \chi'}{2} \left[ \frac{1}{2} \hat{n} \cdot \hat{n}' \right]} e^{-i\psi[n,n'] \frac{1}{2}}$$

$$\psi[n,n'] = \arctan\left[ \frac{\cos(\theta + \theta')}{2} \frac{\tan(\phi - \phi')/2}{\cos(\theta - \theta'}/2}\right]$$

and they provide a resolution of the identity in this two dimensional space:

$$\int \frac{d\hat{n}}{2\pi} |\hat{n}\rangle \langle \hat{n}| = 1$$

where in terms of the angle representation, $d\hat{n} = d\cos \theta d\phi$.

APPENDIX B: COHERENT STATE PATH INTEGRAL FOR QUASIPARTICLES

We develop here a path integral representation for the propagator in the coherent state basis. This propagator is the term within square brackets in equation (73).
To develop its path integral representation, we first make a Trotter-Suzuki expansion: 
\[ e^{-i\hat{H}_{\text{int}}\epsilon} = 1 - i\hat{H}_{\text{int}}\epsilon + O(\epsilon^2), \]
and then insert a complete set of momentum states
\[ |p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ip\cdot r/\hbar} |r\rangle, \]
\[ 1 = \int |p_{j+\frac{1}{2}}\rangle \langle p_{j+\frac{1}{2}}| \]
that yields the identity in wavefunction space. For short time intervals we then obtain:
\[ \mathcal{K}(r_{j+1}, \hat{n}_{j+1}; r_j, \hat{n}_j; \epsilon) \approx \langle \hat{n}_{j+1} | \hat{n}_j \rangle e^{ip_{j+\frac{1}{2}}(r_{j+1}-r_j)/\hbar} e^{-i\hat{\mathcal{H}}(p_{j+\frac{1}{2}}; r_{j+1}, \hat{n}_{j+1})/\hbar} \]
\[ \mathcal{H}(p, r, \hat{n}) = \frac{p^2 + e^2 A^2(r)}{2m} - \mu n_x - \frac{e}{m}[p \cdot A(r)] 
+ \Delta_R(r)n_x + \Delta_{\text{im}}(r)n_y \]
the overlap \( \langle \hat{n}_{j+1} | \hat{n}_j \rangle \) is evaluated using (A-1) and the assumption that the angle between the two unit vectors involved are small, which yields:
\[ \langle \hat{n}_{j+1} | \hat{n}_j \rangle \approx e^{-\frac{1}{2}\cos\theta_j(\phi_{j+1}-\phi_j)} \]
where the arbitrary phase \( \chi \) has been dropped. Putting this all together, we obtain the coherent state path integral representation of the propagator
\[ \mathcal{K}(r', \hat{n}'; r, \hat{n}; t) = \int D\hat{n} D\hat{p} e^{iS/\hbar} \]
where
\[ S = S_B - \int dt \mathcal{H}(r, p, \hat{n}) \]
\[ S_B = \int dt \{ p \cdot \frac{dr}{dt} + \hbar \frac{\hat{n}}{2}[1 - \cos(\theta(t)\frac{d\phi}{dt})] \}

APPENDIX C: THE QUANTIZATION CONDITION

Since we are searching for a quantization condition that will give us the energy eigenstates, it is more convenient to work with the Green function \( \hat{G}(E) \) which depends on the energy and is a Fourier transform of the propagator in time.
\[ \hat{G}(E) = -i \int_0^\infty dt e^{iEt} \hat{K}(t) \]
now the discrete energy eigenstates appear as poles in the Green function. The simplest quantity to calculate that still retains information of these poles is the trace of \( \hat{G}(E) \). Thus, carrying out the trace using spin coherent states we have:

\[ \text{Tr}[\hat{G}(E)] = -i \int_0^\infty dt e^{iEt} \int \frac{d\hat{n}}{2\pi} d^4r \langle \hat{n} r | e^{-i\hat{H}t} | \hat{n} r \rangle \]

(C-2)

Assuming that the classical path makes the dominant contribution to the propagator, we can write
\[ \langle \hat{n} r | e^{-i\hat{H}t} | \hat{n} r \rangle \approx e^{iS_{\text{cl}}(r, \hat{n} t)} \]

(C-3)
where \( S_{\text{cl}}(r, \hat{n} t) \) is the value of the action \( S \), evaluated along the classical path that begins and ends at \( (r, \hat{n}) \) after a time interval \( t_{\text{cl}} \). For definiteness we write out this expression in the London gauge:
\[ \text{(B-1)} \]
\[ \text{Tr}[\hat{G}(E)] = -i \int_0^\infty dt \int d\hat{n} d^4r \langle \hat{n} r | e^{-i\hat{H}t} | \hat{n} r \rangle \]

with \( r_{\text{cl}}(0) = r \) and \( n_{\text{cl}}(0) = \hat{n} \).

Using the approximation (C-3), the integral over \( r \) in (C-2) can be carried out within the stationary phase approximation. This yields:
\[ \frac{\partial S_{\text{cl}}(r, \hat{n} t)}{\partial r} = 0 \Rightarrow p_{\text{cl}}(t) = p_{\text{cl}}(0) \]

(C-4)

Hence we will be interested in classical orbits that return to their beginning phase space point in a time \( t \). The integration over time can be done in a similar fashion; the stationary phase condition gives an equation for the duration \( (t_{\text{E}}) \) of the periodic classical motion:
\[ E = -\frac{\partial S_{\text{cl}}}{\partial t} |_{t_{\text{E}}} = \mathcal{H}_L(r_{\text{cl}}, p_{\text{cl}}, \hat{n}_{\text{cl}}) \]

(C-5)
that is, only those classical paths are selected which have energy \( E \), the argument of the Green’s function. Putting all this together we have:
\[ \text{Tr}[\hat{G}(E)] \sim \sum_{\text{p.c.o.}(E)} e^{iS_{\text{cl}}(r_{\text{cl}}, p_{\text{cl}}, \hat{n}_{\text{cl}})} \]

(C-6)
where the sum runs over all periodic classical orbits with energy \( E \), and
\[ S_{\text{cl}}(r_{\text{cl}}, p_{\text{cl}}, \hat{n}_{\text{cl}}) = S_{\text{cl}} + Et \]
\[ = \int p_{\text{cl}} \cdot dr_{\text{cl}} + \frac{\hbar}{2} \oint A(\hat{n}_{\text{cl}}) \cdot d\hat{n}_{\text{cl}} \]
\[ + \frac{\hbar}{2} \oint d\Phi(r_{\text{cl}}) \]

(C-7)
is the total phase acquired by the quasiparticles along the closed classical trajectory. Since these closed orbits repeat themselves, we can sum over these repetitions explicitly:
\[ \text{Tr}[\hat{G}(E)] \sim \sum_{\text{p.c.o.}(E)} \frac{e^{iS_{\text{cl}}(r_{\text{cl}}, p_{\text{cl}}, \hat{n}_{\text{cl}})}}{1 - e^{iS_{\text{cl}}(r_{\text{cl}}, p_{\text{cl}}, \hat{n}_{\text{cl}})}} \]

(C-8)
where the sum is now evaluated only over a single traversal of the periodic classical orbit. Clearly, there is a pole in the Green’s function if the phase accumulated on this orbit is a multiple of $2\pi$. That is, if:

$$S_{B,c}(r_{cl},p_{cl},\hat{n}_{cl}) = \oint p_{cl} \cdot dr_{cl} + \frac{n}{2} \oint A(\hat{n}_{cl}) \cdot d\hat{n}_{cl}$$

$$+ \frac{\hbar}{2} \oint d\Phi(r_{cl}) = 2\pi \hbar$$

This then is the quantization rule for obtaining the spectrum of the quasiparticle bound states. The simplified discussion here ignores the presence of conjugate points and Maslov indices, which is permissible for the cases considered in this article.

15. For an interesting account of the experimental discovery of the superfluid phases of He$_3$ see D. Osheroff, Rev. Mod. Phys. 69, 667, (1997).
27. I. J. Lee et al. cond-mat/0001332.
35. for a review see: M. Sigrist et al. cond-mat/9902214.
37. For a review see: N. Read, cond-mat/0011338.
39. We are grateful to L. Balents for detailed discussions on this topic.
40. D. A. Ivanov cond-mat/0005069.
42. A. Vishwanath and F.D.M. Haldane, in preparation.

60 For a review see R. Jalabert, ‘The semiclassical tool in mesoscopic physics’ cond-mat/9912038.


63 This notation differs from that used in standard text64 where the symbol $R_d$ is used instead.