Dynamical Systems, Computer Lab 3 IB c205 10/14/2004
Local Stability Analysis of a System of Non-Linear Differential Equations

Starting with a system of n non-linear differential equations:

$$\frac{dX_1}{dt} = f_1(X_1, X_2, \cdots, .Xn)$$
$$\frac{dX_2}{dt} = f_2(X_1, X_2, \cdots, .Xn)$$
$$\vdots$$
$$\frac{dX_n}{dt} = f_n(X_1, X_2, \cdots, .Xn)$$

Calculate the equilibria: X_1^*, X_2^*, …, X_n^* by setting $dX_1/dt = dX_2/dt = \ldots = dX_n/dt = 0$.

Define a small perturbation from equilibrium:

$X_1 = X_1^* + x_1$
$X_2 = X_2^* + x_2$
$\vdots$
$X_n = X_n^* + x_n$

Use Taylor's Theorem to linearize the differential equation near equilibrium:

$$\frac{dx_1}{dt} \cong \left.\frac{\partial f_1}{\partial X_1}\right|_* x_1 + \left.\frac{\partial f_1}{\partial X_2}\right|_* x_2 + \cdots + \left.\frac{\partial f_1}{\partial X_n}\right|_* x_n$$
$$\frac{dx_2}{dt} \cong \left.\frac{\partial f_2}{\partial X_1}\right|_* x_1 + \left.\frac{\partial f_2}{\partial X_2}\right|_* x_2 + \cdots + \left.\frac{\partial f_2}{\partial X_n}\right|_* x_n$$
$$\vdots$$
$$\frac{dx_n}{dt} \cong \left.\frac{\partial f_n}{\partial X_1}\right|_* x_1 + \left.\frac{\partial f_n}{\partial X_2}\right|_* x_2 + \cdots + \left.\frac{\partial f_n}{\partial X_n}\right|_* x_n$$

This is a set of linear constant coefficient equations, which can be written in the form:

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} \left.\frac{\partial f_1}{\partial X_1}\right|_* & \left.\frac{\partial f_1}{\partial X_2}\right|_* & \cdots & \left.\frac{\partial f_1}{\partial X_n}\right|_* \\ \left.\frac{\partial f_2}{\partial X_1}\right|_* & \left.\frac{\partial f_2}{\partial X_2}\right|_* & \cdots & \left.\frac{\partial f_2}{\partial X_n}\right|_* \\ \vdots & & & \\ \left.\frac{\partial f_n}{\partial X_1}\right|_* & \left.\frac{\partial f_n}{\partial X_2}\right|_* & \cdots & \left.\frac{\partial f_n}{\partial X_n}\right|_* \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

x'(t) = **A x**(t), where **A** is the Jacobean matrix.

The equilibrium will be locally-stable if the real part of all of the eigenvalues of the A are less than zero.

Local Stability analysis for a systems of 2 non-linear differential equations.

We can look at the solution in general for systems of 2 differential equations.
We will get the Jacobean matrix:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

Det($\mathbf{A} - \lambda \mathbf{I}$)=0

$(A_{11} - \lambda)(A_{22} - \lambda) - A_{12}A_{21} = 0$

$\lambda^2 - \lambda(A_{11} + A_{22}) + (A_{11}A_{22} - A_{12}A_{21}) = 0$

The solution to this is:

$$\lambda = \frac{A_{11} + A_{22}}{2} \pm \frac{\sqrt{(A_{11} + A_{22})^2 - 4(A_{11}A_{22} - A_{12}A_{21})}}{2}$$

For local stability, we need the real parts of all eigenvalues to be negative. In order for this to be met, the following conditions must be met:

(1) $(A_{11} + A_{22}) < 0$
(2) $A_{11}A_{22} > A_{12}A_{21}$

These are the **Routh-Hurwitz** Criteria for stability of a 2-species interaction.
They are a short-hand way of telling whether an equilibrium is locally stable or not.

Routh-Hurwitz Criteria for higher order systems.
There are equivalents for the Routh-Hurwitz Criteria for third and fourth order systems:
Another way of writing the criteria for the second order system is:
If you have the characteristic equation:
$\lambda^2 + a_1 \lambda + a_2 = 0$
For local stability: $a_1 > 0$, $a_2 > 0$

For a third order system:
$\lambda^3 + a_1\lambda^2 + a_2 \lambda + a_3 = 0$
For local stability: $a_1 > 0$, $a_3 > 0$, $a_1a_2 > a_3$

For a fourth order system:
$\lambda^4 + a_1 \lambda^3 + a_2\lambda^2 + a_3 \lambda + a_4 = 0$
For local stability: $a_1 > 0$, $a_3 > 0$, $a_3 > 0$, $a_1a_2a_3 > a_3^2 + a_1^2a_4$

For systems higher than this they become unwieldy.

Simple Harmonic Motion

Consider the linear second order differential equation, which describes the movement of a mass on a spring:

$$\frac{d^2x}{dt^2} + Kx = 0$$

where x is the displacement of the mass from the spring's un-stretched position. The solution to this equation was simple harmonic motion, cycles with constant amplitude determined by the initial displacement of the mass.

A second order differential equation can be converted to 2 first order differential equations, which we can solve in Matlab:

Define a new variable $y = \frac{dx}{dt}$ (in this case y represents the velocity of the mass).

Then $\frac{dy}{dt} = \frac{d^2x}{dt^2}$, and the original equation can be re-written as $\frac{dy}{dt} + Kx = 0$.

We now have two first order equations:

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -Kx$$

Homework task 1:

Use Matlab to plot numerical solutions to this system of equations for K= 1. Explore the effect of changing the initial conditions. Plot x vs. time, and x vs. y.

Neutrally-Stable Cycles

Homework task 2:

Use Matlab to obtain numerical solutions for the non-linear Lotka-Volterra predator-prey model:

Prey: $\frac{dN}{dt} = rN - aNP$

Predator: $\frac{dP}{dt} = caNP - mP$

Plot N versus P, and try varying the initial conditions. The amplitude of the cycles depends on the initial conditions, but now the cycles are orbiting the equilibrium predator and prey densities, rather than zero. These are **neutrally-stable cycles**. We showed in class that the eigenvalues of the Jacobean matrix for this system are $\lambda = \pm i\sqrt{rm}$.

Homework task 3:

Perform local stability analysis on the modification of the Lotka-Volterra predator-prey model with logistic prey growth, where r is the per-capita growth rate at low densities, and K is the carrying capacity of prey:

Prey: $$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - aNP$$

Predator: $$\frac{dP}{dt} = caNP - mP$$

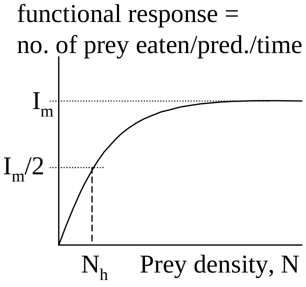
Limit Cycles

Now consider this more complicated **predator-prey model**:

prey: $$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - \left(\frac{I_m N}{N + N_H}\right)P$$

predator: $$\frac{dP}{dt} = \left(\frac{eI_m N}{N + N_H}\right)P - mP$$

In this model, the prey growth rate is logistic and the predator eats the prey according to a saturating, Type II, functional response, in which I_m is the maximum ingestion rate per predator, and N_H is the half-saturation prey density (i.e. the density of prey at which predators have an ingestion rate of half of their maximum value). Prey eaten by predators are converted to new predators, with a conversion efficiency of *e*. Predators have a density-independent per-capita death rate of *m*.



Optional Homework task 4:

Perform numerical simulations in Matlab to show that for different values of *K*, this system can have either a stable equilibrium or can produce cycles. The amplitude of the cycles no longer depends on the initial conditions. These are **stable limit cycles**. The system returns to the same cyclic attractor, regardless of the initial conditions.