

Lecture XII

One topic toward the end of the last lecture was the excitation spectrum of superfluids and superconductors. Our conclusion was that in superfluids, there is an ordinary gapless “phonon” mode (note that here, as in common usage, one refers to sound quanta in helium as phonons, even though there is no lattice involved), while in superconductors, the corresponding compressional mode of electrons gets pushed up to the plasma frequency through the coupling to an electromagnetic gauge field: through the “Higgs mechanism” both the gapless photon and gapless phonon disappear.

How is a superflow consistent with the existence of gapless phonon excitations? We can see from a simple calculation how it may be unfavorable to create phonons to degrade a superflow, even though phonon excitations exist at arbitrarily small energy. Suppose we are at $T = 0$, so that all atoms are in the superfluid (just as at $T = 0$ all electrons are in the superconductor) in a two-fluid model. How fast can the superfluid flow through a tube before its velocity is degraded by the creation of excitations above the ground state? Suppose the superfluid moves with velocity \mathbf{v} relative to the walls of the tube and contains a single elementary excitation of momentum \mathbf{p} and energy ϵ . Then in the frame where the superfluid is at rest, its energy is $E_0 = \epsilon$ and momentum $\mathbf{P}_0 = \mathbf{p}$ (here energy is defined so that the ground state has energy 0). Suppose also that its total mass is M . Using Galilean invariance, the energy and momentum in the laboratory frame are

$$E = E_0 + \mathbf{P}_0 \cdot \mathbf{v} + \frac{1}{2}Mv^2 = \epsilon + \mathbf{p} \cdot \mathbf{v} + \frac{1}{2}Mv^2, \quad \mathbf{P} = \mathbf{P}_0 + M\mathbf{v} = \mathbf{p} + M\mathbf{v}. \quad (1)$$

For the excitation to lower the energy of the system relative to the moving ground state with no excitation, it must satisfy $\epsilon + \mathbf{p} \cdot \mathbf{v} < 0$; it is easiest to create excitations when \mathbf{p} and \mathbf{v} are opposite. Then the required velocity to generate excitations is

$$v > \min(\epsilon/p). \quad (2)$$

Here the minimum value is taken over all possible elementary excitations. Hence it is crucial for a nonzero critical velocity that the elementary excitation have a linear dispersion relation near zero momentum, like a phonon; if the excitations were quadratic in momentum, then the critical velocity would be zero. In this context it is important that the phonon branch of the excitation spectrum $\epsilon(p)$ of He^4 is nonmonotonic: there is a secondary minimum at finite p known as the “roton” minimum. Rotons are quasiparticles near this dip in $\epsilon(p)$, but one should keep in mind that rotors and phonons are really derived from the same physics.

Note that in the above we scarcely had to use the $T = 0$ assumption that the system is in its ground state. At finite temperature, helium hydrodynamics can be modeled as consisting of a superfluid part and a normal part, just as the Ginzburg-Landau equation in a superconductor is normally taken to describe just the superconducting condensate of electrons. A generalization of the above to finite temperature is also given in Landau/Lifshitz volume 9 and also Khalatnikov.

Penrose-Yang definition of ODLRO.

A final comment about superfluids: there is an amazing sort of elementary excitation which is actually important in understanding how real superfluid flows degrade. Suppose that, instead of being straight, a vortex line is bent around into a vortex ring. It can be shown, from a classical

hydrodynamic calculation, that such a vortex ring has a velocity determined by its size: $v \approx (\hbar/2mR_0) \log(R_0/a)$, where a is some atomic length scale and R_0 is the size of the ring. In a small capillary, the observable critical velocity is sometimes limited by the creation of vortex rings, rather than by the creation of quasiparticles as above.

Many of the amazing features of superfluid helium result from its translational invariance. This explains a large part of why superfluid helium is the system of choice for precise measurements of phase transitions; the other part is the absence of long-range interactions. Our next topic, the fractional quantum Hall effect, depends on breaking of translational invariance, and not just by a regular lattice: in some sense, “dirt” (a random potential with no periodicity) is necessary to explain both quantum Hall effects. First note that any Lorentz-invariant 2D system in perpendicular \mathbf{E} and \mathbf{B} fields must have a very simple Hall coefficient

$$R_H = \frac{V_y}{BI_x} = -\frac{1}{nec}. \quad (3)$$

Here n is the density of charge carriers. The above results from a current density

$$j = -nec \frac{\mathbf{E} \times \mathbf{B}}{B^2}, \quad (4)$$

so $I_x = jL$, $E_y = V_y/L$. Then in the end

$$R_H = -\frac{1}{nec} \quad (5)$$

The above current density is forced by Lorentz invariance since for $E < B$ we can carry out a Lorentz transformation to a frame where E is zero; the velocity of this frame gives the above current density.

The integer quantum Hall effect (the quantization of the transverse conductance in units of $\frac{e^2}{h}$) depends on the existence of disorder so that the above Lorentz-invariance argument is violated; for the noninteracting case, it is simple to show that even a periodic potential is insufficient to give the correct behavior, and some sort of disorder is required. We will now discuss the physics of localization by disorder in zero magnetic field, both as an interesting topic in its own right and as a prelude to the discussion of the IQHE and FQHE.

Consider the solutions of Schrodinger’s equation for a single electron moving in a random potential. This problem may seem to have little to do with the sort of interacting clean problems we discussed previously, but actually the same field-theoretic methods are useful for both. Instead we will be content to combine some simple arguments in general dimensionality with a simple calculation due to Halperin in one dimension. The surprising feature is that even a weak disorder potential can lead to localization of electronic eigenstates: the early steps by Anderson and Mott in understanding this phenomenon were rewarded by the 1977 Nobel Prize.

Our simple picture for the behavior of eigenstates in a random potential would probably go as follows. At low energy, there should be some bound states near minima of the potential, while at high energy, we expect that there should be some free states, where the electron scatters occasionally off bumps in the potential but is unbound. A mathematical distinction can be made between “localized” eigenstates whose magnitude falls off exponentially at spatial infinity, and “extended” eigenstates which fall off more slowly. (Often the term “critical” is used for wavefunctions that fall off algebraically, and “extended” reserved for wavefunctions like plane waves that do not fall off at all.)

There is a simple argument due to Mott that shows that extended and localized states should not both exist at the same energy in a generic random potential. Therefore there is a “mobility edge”: there is a particular energy E_c above which eigenstates are extended, and below which eigenstates are localized. The argument is that if both extended and localized states were present at the same energy, then because the energy denominator is zero, even a small perturbation would strongly mix the extended and localized states, giving two extended states.

Actually, a more mathematically useful way to proceed with this argument is to consider time evolution of a wavepacket, rather than energy eigenstates. Following the original 1958 paper of Anderson, consider the following tight-binding lattice model for single particles:

$$H = \sum_i U_i n_i - t \sum_{\langle ij \rangle} (c_i^\dagger c_j + \text{h.c.}). \quad (6)$$

Here the U_i are some random variables sampled from a distribution whose details are relatively unimportant, as long as it falls off exponentially at energies far away from some central value U_0 .

Suppose we start with one electron in the localized state at the origin O : $\psi(0) = c_O^\dagger |0\rangle$. This state is not an eigenstate because of the hopping operator proportional to t , so over time the electron density spreads out. If there are some extended states, then one expects that at sufficiently long times, the density spread will be diffusive with some diffusion constant D :

$$\langle R^2 \rangle = \int |\psi(x)|^2 R^2 d^d x \sim Dt. \quad (7)$$

However, if all the states in the system are sharply localized, then at long times the density will have ceased to spread:

$$\lim_{t \rightarrow \infty} \langle R^2 \rangle = \xi^2. \quad (8)$$

Here ξ is some quantity with units of length, referred to as the “localization length.”

It is perhaps worth pointing out that these are not the only two alternatives. It was understood only quite recently that if one adds a magnetic field and works within the lowest Landau level instead of using a lattice model, then $\langle R^2 \rangle \sim t^\alpha$ for some number $\alpha \approx 0.79$, but a calculable understanding of this number is still lacking.

Going back to the question of whether there is a mobility edge between extended and localized states: this picture is essentially correct in three dimensions, and there is a “localization transition” as the Fermi level moves through the mobility edge. However, in one and two dimensions, an amazing sort of quantum interference leads to localization of eigenstates at **all** energies, even for a weak potential. We will see this explicitly in one dimension in the next lecture. Why is dimensionality so important, given that the above mobility edge picture didn’t seem to depend on dimensionality?

Well, it is fairly easy to argue that extended states should survive above two dimensions. Consider diffusive propagation of electrons, as we would expect if they are scattering occasionally off potential fluctuations, without becoming localized. (Eigenstates can be “ballistic”, like plane waves, “diffusive”, like these scattering states, or “localized”.) Diffusive spreading means $\langle R^2 \rangle \sim Dt$: we can think of the electron density at time t as concentrated in a sphere of radius proportional to \sqrt{Dt} . (More realistically, of course, the probability distribution of the density would be Gaussian.) Then, normalizing the overall density to 1, we have that the probability for the particle to be near the origin at time t goes as $(Dt)^{-d/2}$, where d is the spatial dimensionality, since this is the

reciprocal of the sphere's volume. Now we can ask, how many times is the electron expected to have returned to the origin by time T ? The expected number of returns is

$$N = \int_{t_0}^T \frac{1}{(Dt)^{d/2}} dt. \quad (9)$$

Here we ignore any possible singularity at the origin (since we know that at short times our assumption of a Gaussian spread breaks down) and focus on the long-time behavior. The integral converges for $d > 2$. So for $d > 2$, the electron returns only a finite number of times to any particular fluctuation; if the fluctuations are weak enough, then the electron will not be localized since different fluctuations are independent for a random potential with only short-ranged correlations.

What about in dimensionality 2 or below? Then a diffusive electron would see some potential fluctuations over and over again, which creates the possibility of constructive or destructive interference of electron waves. Note that localization depends crucially on phase coherence; destruction of phase coherence by inelastic scattering (off of phonons, for instance) leads to delocalization. An argument that constructive interference leads to localization is as follows: consider two paths in the Feynman path integral over classical histories for Ψ which differ in that one interior loop is traversed in one direction (say clockwise) by the first path, and in the opposite direction by the second path. In zero magnetic field, for any realization these two contribute equally. We have, writing ψ_1 and ψ_2 for the contributions from the two paths, a sort of constructive interference for such paths:

$$\langle |\Psi|^2 \rangle = \langle |\psi_1 + \psi_2|^2 \rangle = \langle |2\psi_1|^2 \rangle = 4\langle |\psi_1|^2 \rangle > \langle |\psi_1|^2 \rangle + \langle |\psi_2|^2 \rangle. \quad (10)$$

This implies that paths with more self-intersections have relatively greater probability than paths with fewer self-intersections. Since total probability is conserved, this means that the electron is likely to stay close to the origin, since paths with many self-intersections tend to be less spatially extended. A detailed calculation in two dimensions that such interference leads to localization of all eigenstates is quite complicated; here we will be content to show localization in 1D.

The term “weak localization” is used to describe perturbative calculations (perturbative in the strength of disorder) of incipient localization; then the same Green's functions techniques can be used as introduced before, although one has to calculate the two-body rather than the one-body Green's function.