Honors undergraduate seminar, Spring 2004
Introduction to emergent properties of many-particle systems
Lecture 11: course webpage at http://socrates.berkeley.edu/˜jemoore

We start off by extending the rescaling calculation at the end of lecture 10 to a case where there is a nontrivial fixed point under rescaling. We will study the two-dimensional rather than the one-dimensional Ising model, and will need to make a bad approximation right at the beginning, but the basic idea of the result is correct for $d \geq 2$. Then we will discuss (at a more qualitative level) some important recent developments in the areas of symmetries and scale invariance.

Rescaling transformations in $d \geq 2$ are more elegant without a lattice, but for simplicity we consider an approximation to the 2D Ising model. This approximation is known as the "Migdal-Kadanoff approximation": its advantage is its simplicity, but it is not much used in practice since it is an uncontrolled approximation. The problem with real-space renormalization on the 2D Ising model is that we tend to generate lots of nonlocal couplings. One advantage of continuum methods (discussed in a graduate course on field theory or statistical mechanics) is that, by working in the basis of scaling variables, it is much more clear which perturbations are relevant and which are not. Migdal-Kadanoff is a trick to redesign the 2D lattice so that the RG equations are closed, without generating nonlocal couplings.

More precisely, to make the Migdal-Kadanoff approximation correct, one should work on a "hierarchical lattice" that is not continuously connected to normal lattices like the square lattice. While there are still papers using Migdal-Kadanoff, a much preferable set of approaches depend on controlled expansions where the small parameter may be related to dimensionality ($\epsilon-$expansion) or the symmetry of the order parameter (large-N expansion), to cite two famous examples.

The rescaling procedure in this approximation consists of two steps: 1. bond moving. 2. decimation. (A picture of this, in case you've missed class, is in Huang's stat mech book problem 18.2, although there are some typos in that discussion.) The rescaling equation is

$$e^{2K'} = \cosh(4K), \tag{1}$$

where before, for the 1D model, it was,

$$e^{2K'} = \cosh(2K), \tag{2}$$

In terms of $x = e^{2K}$, the Migdal-Kadanoff RG equation is

$$x' = \frac{x^2 + x^{-2}}{2}. \tag{3}$$

In addition to the $x = 1$ and $x = \infty$ solutions that were present before, this equation has an intermediate fixed point. Finding that point x^* exactly requires solution of a quartic, which we won't bother to do (can be done in *Mathematica*), but it is useful to understand a bit more about this fixed point. The easiest way to show that there must be an unstable fixed point between 1 and ∞ is to plot the function $f(x) = -x + (x^2 + x^{-2})/2$.

One reason why rescaling or "renormalization group" methods like the above have revolutionized particle physics and condensed matter physics is that physical singularities, like those that exist near a critical point, become just fixed points of a smooth map, the rescaling transformation. Our last topic in this area (next lecture) will be to say a bit more about such rescaling transformations as a justification of "universality".

Symmetries and scale invariance:

We can understand the result of the above as saying that critical points have a very special behavior under rescaling: in some sense they are "scale-invariant." A natural question is whether behavior under scaling transformations can be viewed in the same way as under other spacetime transformations, like rotations. Since rotational symmetry generates so much interesting physics (angular momentum, spin, etc.), let's take a quick look at symmetry under **dilations** like $\mathbf{x} \to \alpha\mathbf{x}$.

Note that most symmetries in physics can be described as either "internal" symmetries or "spacetime" symmetries: the former includes gauge symmetries that lead to conserved charges, while spacetime symmetries include the Poincare group (made up of translations and Lorentz transformations). A very fundamental fact of physics is that the representations of the Poincare group for massive or massless particles determine the allowed spin and momentum values. Modern physics has seen huge developments in both internal and spacetime symmetries, and we will discuss the gauge symmetries of the Standard Model briefly below. We will even discuss a tiny bit of string theory.

Let us play around with extending the Poincare group to include dilations. The group we obtain will be known as the "global conformal group": these are linear transformations of spacetime that preserve angles, but not distances. We are also going to work in Euclidean space rather than Minkowski space, so the starting metric is just $g_{\mu\nu} = \delta_{\mu\nu}$. We look for transformations that take $\mathbf{x}$ to $\mathbf{x}'$ and have

$$g'_{\mu\nu}(\mathbf{x}') = \Lambda(\mathbf{x}) g_{\mu\nu}(\mathbf{x}). \tag{4}$$

This says that the metric changes by a locally variable scale factor, but the overall x is constant.

To get the dimensionality of the global conformal group, it suffices to look at infinitesimal transformations: as an example, an infinitesimal rotation by angles ϵ^i around the origin is

$$x^\alpha \to x^\alpha + \epsilon^i L^{i\alpha}_\beta x^\beta \tag{5}$$

where $L^{i\alpha}_\beta$ is one of the $d(d-1)/2$ generators of rotations, which are antisymmetric real matrices. There is one pure dilation from the origin, and d pure translations. Are there any other generators? Under an infinitesimal transformation

$$\mathbf{x} \to \mathbf{x}' = \epsilon(\mathbf{x}) + \mathbf{x}, \tag{6}$$

the induced change in the metric is

$$g_{\mu\nu} \to g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} = g_{\alpha\beta} \frac{1}{(\delta^\mu_\alpha + \partial_\alpha \epsilon^\mu)(\delta^\nu_\beta + \partial_\beta \epsilon^\nu)} = g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu). \tag{7}$$

The derivative term is zero for pure rotations or translations. The requirement of conformal invariance can be stated as

$$f(\mathbf{x}) g_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu. \tag{8}$$

Taking the trace of both sides gives

$$f(\mathbf{x}) d = 2 \partial_\mu \epsilon^\mu. \tag{9}$$

Now we will combine the above into a full description of conformal transformations. Taking ∂_ρ of (5) gives

$$g_{\mu\nu} \partial_\rho f = \partial_{\rho\mu} \epsilon_\nu + \partial_{\rho\nu} \epsilon_\mu. \tag{10}$$

Subtract this from two copies of the same formula (with permuted indices $\nu \leftrightarrow \rho$ for one copy, and $\mu \leftrightarrow \rho$ for the other copy) to get

$$g_{\nu\rho}\partial_\mu f + g_{\mu\rho}\partial_\nu f - g_{\mu\nu}\partial_\rho f = 2\partial_{\mu\nu}\epsilon_\rho. \tag{11}$$

Multiplying by $g^{\mu\nu}$ and summing these two indices gives

$$2\partial^2\epsilon_\rho = (2-d)\partial_\rho f. \tag{12}$$

So for $d = 2$ any transformation that makes the left-hand-side 0 will be conformal.

For $d > 2$, one can work from this equation and show that there are d "special conformal transformations" or SCTs: the global conformal group is generated by 1 dilation, d translations, $d(d-1)/2$ rotations, and d SCTs, so the total dimension is $(d+1)(d+2)/2$. The SCTs look like a combination of an inversion, a translation, then another inversion:

From the above equations, one can see that there might be something special about two dimensions. In two dimensions, any transformation that is "complex analytic" preserves angles and hence is conformally invariant, and we return to this below. In one dimension, any smooth transformation is conformal but there is no notion of angle, so this is somewhat trivial.

If you have never taken a course in complex analysis, you can think of a complex analytic function of $z = x + iy$ as one that has a Taylor series expansion in z without any appearances of the conjugate $\bar{z}$. A simple example of a nonanalytic function is $\mathrm{Re}(z)$. The group of mappings that are one-to-one from the plane to itself has dimension 6, just as one would expect from the above dimension counting: these transformations can be taken of the form

$$z \to \frac{az+b}{cz+d} \quad ad - bc = 1. \tag{13}$$

If you like, you can show that these transformations form a group of dimension six by noting that composing two transformations is like multiplying two two-by-two complex matrices.

There are a huge number of functions that are complex analytic but either not defined on the whole plane, or not globally one-to-one. It turns out that scale-invariant theories are invariant not only under global transformations but also under these local transformations, which form an enormous symmetry group (infinite-dimensional). As a result, since 1984 physicists have learned how to solve and classify a very large number of two-dimensional theories. The primary motivation for learning about these theories came not from 2D stat mech but from string theory. Note that the worldsheet of a string is two-dimensional: the string traces out a surface in spacetime. In 4D spacetime, four functions of one variable can be used to describe the 4D trajectory of a particle: $\mathbf{x}^\mu(\tau)$ gives the space and time coordinates as a function of some parameter τ along the trajectory. In the same way, d functions of two variables describe the path of a string.

You have probably heard statements like "string theory is only consistent in 10 or 26 dimensions." It turns out that, for a string theory with only bosons, the requirement of conformal invariance leads to a subtle quantum pathology, known as the conformal anomaly, except in dimension 26! With fermions and bosons, there is an anomaly except in dimension 10. So conformal invariance is crucial in understanding which dimensions support viable string theories. Another modern breakthrough in symmetry was the discovery of "supersymmetries" that expand the Poincare group to include extra fermionic generators.

Finally, the internal symmetries that appear in the Standard Model have been studied using the rescaling (or "renormalization group") approach that we mentioned above in the context of critical points. You have probably heard that the standard model gauge group is $U(1)\times SU(2)\times SU(3)$, and that the quantum chromodynamics part (the $SU(3)$) behaves very differently from the electroweak part.

QCD is a confining theory at low energy or large distance: quarks are bound together into hadrons and mesons. The famous calculation of Gross and Wilczek (done indepenently by Politzer) to understand this behavior was a renormalization group calculation like the above. They showed that for $SU(3)$ gauge theory with sufficiently few families of fermions, the coupling constant increases at low energy or long length scale, while at high energy, the coupling constant weakens. Theories with vanishing coupling at high energy, like QCD, are known as "asymptotically free", and this hypothesis of asymptotic freedom has been confirmed in collider experiments.